*Journal of Nepal Mathematical Society (JNMS)*
*ISSN: 2616-0153 (Print), 2616-0161 (Online)*
*Vol. 8, Issue 2, 2025 (Dec.): 67-72*
*DOI:* https://doi.org/10.3126/jnms.v8i2.88068


**Research Article**
*Received Date: Nov. 02, 2025*
*Accepted Date: Dec. 14, 2025*
*Published Date: Dec. 29, 2025*

# A New Derivation of Extended Watson Summation Theorem with an Application

Resham Prasad Paudel[1], Harsh Vardan Harsh[2,*], Madhav Prasad Poudel[3], Ganesh Bahadur Basnet[1]

[1]Department of Mathematics, Tribhuvan University, Tri-Chandra Multiple Campus, Kathmandu, Nepal
[2]Parul Institute of Technology, Parul University, Vadodhara, Limda, Gujarat, India
[3] School of Engineering, Pokhara University, Pokhara, Nepal

*Correspondence to: Harsh Vardan Harsh, Email: harshvardhanharsh@gmail.com

**Abstract:** *In this research article, we aim to provide a new proof of the extended Watson summation theorem for the series ${}_4F_3$ obtained earlier by Kim et al. As an application, we evaluate four interesting integrals involving generalised hypergeometric function.*

**Keywords:** Generalized hypergeometric function, Watsen theorem, Extended Watson theorem, Gauss theorem.

## 1 Introduction

The classical summation theorems for the generalized hypergeometric series such as those of Gauss, Gauss second, Kummer and Bailey for the series ${}_2F_1$; Watson, Dixon and Whipple for the series ${}_3F_2$ play a key role in theory and application. Bailey pointed out several interesting applications by using the above mentioned summation theorems [1].

In 2010, these summation theorems were extended by Kim et al. [5]. Here we would like to mention some of the results that will be required in our present investigation.

**Extension of Gauss second summation theorem** [5]

$$\begin{aligned} {}_3F_2\left[\begin{matrix}\alpha,\beta,\delta+1\\ \frac{1}{2}(\alpha+\beta+3),\delta\end{matrix};\frac{1}{2}\right] &= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}(\alpha+\beta+3)\right)\Gamma\left(\frac{1}{2}(\alpha-\beta-1)\right)}{\Gamma\left(\frac{1}{2}(\alpha-\beta-3)\right)} \\ &\times\left\{\frac{\frac{1}{2}(\alpha+\beta-1)-\frac{\alpha\beta}{\delta}}{\Gamma\left(\frac{1}{2}(\alpha+1)\right)\Gamma\left(\frac{1}{2}(\beta+1)\right)}+\frac{\frac{\alpha+\beta+1}{\delta}-2}{\Gamma\left(\frac{1}{2}\alpha\right)\Gamma\left(\frac{1}{2}\beta\right)}\right\} \end{aligned} \tag{1.1}$$

For $\delta=\frac{1}{2}(\alpha+\beta+1)$, (1.1) reduces to the well-known Gauss second summation theorem [2, 8] viz.

$${}_2F_1\left[\begin{matrix}\alpha,\beta\\ \frac{1}{2}(\alpha+\beta+1)\end{matrix};\frac{1}{2}\right]=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}(\alpha+\beta+1)\right)}{\Gamma\left(\frac{1}{2}(\alpha+1)\right)\Gamma\left(\frac{1}{2}(\beta+1)\right)} \tag{1.2}$$

**Extension of Watson summation theorem** [5]

$$\begin{aligned} &{}_4F_3\left[\begin{matrix}\alpha,\beta,\gamma,\delta+1\\ \frac{1}{2}(\alpha+\beta+3),2\gamma,\delta\end{matrix};1\right] \\ &= \frac{2^{\alpha+\beta-2}\Gamma\left(\gamma+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}(\alpha+\beta+3)\right)\Gamma\left(\gamma-\frac{1}{2}(\alpha+\beta+1)\right)}{(\alpha-\beta-1)(\alpha-\beta+1)\Gamma(\frac{1}{2})\Gamma(\alpha)\Gamma(\beta)} \\ &\times\left\{\gamma_1\frac{\Gamma(\frac{1}{2}\alpha)\Gamma(\frac{1}{2}\beta)}{\Gamma\left(\gamma-\frac{1}{2}\alpha+\frac{1}{2}\right)\Gamma\left(\gamma-\frac{1}{2}\beta+\frac{1}{2}\right)}+\gamma_2\frac{\Gamma\left(\frac{1}{2}(\alpha+1)\right)\Gamma\left(\frac{1}{2}(\beta+1)\right)}{\Gamma\left(\gamma-\frac{1}{2}\alpha\right)\Gamma\left(\gamma-\frac{1}{2}\beta\right)}\right\} \\ &= \Omega \end{aligned} \tag{1.3}$$

provided $Re(2\gamma-\alpha-\beta)>-1$.
Also the constants $\gamma_1$ and $\gamma_2$ are given by

$$\gamma_1=\alpha(2\gamma-\alpha)+\beta(2\gamma-\beta)-2\gamma+1-\frac{\alpha\beta}{\delta}(4\gamma-\alpha-\beta-1) \tag{1.4}$$

and

$$\gamma_2=\frac{4}{\delta}(\alpha+\beta+1)-8. \tag{1.5}$$

For $\delta=\frac{1}{2}(\alpha+\beta+1)$, (1.3) reduces to the classical Watson summation theorem [2] viz.

$$\begin{aligned}&{}_3F_2\left[\begin{matrix}\alpha,\beta,\gamma,\\ \frac{1}{2}(\alpha+\beta+1),2\gamma\end{matrix};1\right]\\ &=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\gamma+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}(\alpha+\beta+1)\right)\Gamma\left(\gamma-\frac{1}{2}(\alpha+\beta-1)\right)}{\Gamma\left(\frac{1}{2}(\alpha+1)\right)\Gamma\left(\frac{1}{2}(\beta+1)\right)\Gamma\left(\gamma-\frac{1}{2}\alpha+\frac{1}{2}\right)\Gamma\left(\gamma-\frac{1}{2}\beta+\frac{1}{2}\right)}\end{aligned} \tag{1.6}$$

provided $Re(2\gamma-\alpha-\beta)>-1$.

**Gauss summation theorem** [2, 8]

$${}_2F_1\left[\begin{matrix}\alpha,\beta\\ \gamma\end{matrix};1\right]=\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} \tag{1.7}$$

provided $Re(\gamma-\alpha-\beta)>0$.

**A special case of (1.7)** [8]

$${}_2F_1\left[\begin{matrix}-\frac{1}{2}n,-\frac{1}{2}n+\frac{1}{2}\\ \gamma+\frac{1}{2}\end{matrix};1\right]=\frac{2^n(\gamma)_n}{(2\gamma)_n}. \tag{1.8}$$

It is interesting to mention here that Kim et al. [5] established the result (1.3) by using contiguous functions relations together with Watson summation theorem (1.6) and its contiguous result which was obtained by Lavoie et al. [5].

In 2017, Choi et al. [3] established the result (1.3) by using the results (1.7) and (1.8). In this work, we aim to provide a new proof of the extended Watson summation theorem(1.3).

## 2 Derivation of (1.3)

In order to derive the result (1.3), we proceed as follows :
Consider the integral

$$\int_0^1 t^{\gamma-1}(1-t)^{\gamma-1}{}_3F_2\left[\begin{matrix}\alpha,\beta,\delta+1\\ \frac{1}{2}(\alpha+\beta+3),\delta\end{matrix};t\right]dt$$

for $Re(\gamma)>0$.

Expressing ${}_3F_2$ as a series and changing the order of integration and series which is permitted due to the uniform convergence of the series, we have

$$I=\sum_{n=0}^{\infty}\frac{(\alpha)_n\ (\beta)_n\ (\delta+1)_n}{\left(\frac{1}{2}(\alpha+\beta+3)\right)_n(\delta)_n n!}\ \int_0^1 t^{\gamma+n-1}(1-t)^{\gamma-1}dt.$$

Evaluating the well-known gamma integral and making use of the relation of following Pochhammer symbol with the gamma function

$$(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)}$$

and we have after some simplification

$$I = \frac{\Gamma(\gamma)\Gamma(\gamma)}{\Gamma(2\gamma)} \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n \ (\gamma)_n \ (\delta+1)_n}{\left(\frac{1}{2}(\alpha+\beta+3)\right)_n (2\gamma)_n(\delta)_n n!} \tag{2.1}$$

Summing up the series, we have

$$I = \frac{\Gamma(\gamma)\Gamma(\gamma)}{\Gamma(2\gamma)} {}_4F_3 \left[ \begin{matrix} \alpha, \beta, \gamma, \delta+1 \\ \frac{1}{2}(\alpha+\beta+3), 2\gamma, \delta \end{matrix} ; 1 \right] \tag{2.2}$$

On the other hand, writing (2.1) in the form

$$I = \frac{\Gamma(\gamma)\Gamma(\gamma)}{\Gamma(2\gamma)} \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n \ (\delta+1)_n}{\left(\frac{1}{2}(\alpha+\beta+3)\right)_n (\delta)_n 2^n n!} \left\{ \frac{2^n(\gamma)_n}{(2\gamma)_n} \right\}$$

Now using (1.8), we have

$$I = \frac{\Gamma(\gamma)\Gamma(\gamma)}{\Gamma(2\gamma)} \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n \ (\delta+1)_n}{\left(\frac{1}{2}(\alpha+\beta+3)\right)_n (\delta)_n 2^n n!} {}_2F_1 \left[ \begin{matrix} -\frac{1}{2}n, -\frac{1}{2}n+\frac{1}{2} \\ \gamma+\frac{1}{2} \end{matrix} ; 1 \right]$$

Now, writing ${}_2F_1$ as a series, we have after some calculations

$$I = \frac{\Gamma(\gamma)\Gamma(\gamma)}{\Gamma(2\gamma)} \sum_{n=0}^{\infty} \sum_{m=0}^{\left[\frac{n}{2}\right]} \frac{(\alpha)_n(\beta)_n \ (\delta+1)_n(-\frac{1}{2}n)_m(-\frac{1}{2}n+\frac{1}{2})_m}{\left(\frac{1}{2}(\alpha+\beta+3)\right)_n (\delta)_n 2^n(\gamma+\frac{1}{2})_m m!n!}$$

Using the identity

$$(-n)_{2m} = 2^{2m}\left(-\frac{1}{2}n\right)_m \left(-\frac{1}{2}n+\frac{1}{2}\right)_m = \frac{n!}{(n-2m)!}$$

we have

$$I = \frac{\Gamma(\gamma)\Gamma(\gamma)}{\Gamma(2\gamma)} \sum_{n=0}^{\infty} \sum_{m=0}^{\left[\frac{n}{2}\right]} \frac{(\alpha)_n(\beta)_n \ (\delta+1)_n}{\left(\frac{1}{2}(\alpha+\beta+3)\right)_n (\gamma+\frac{1}{2})_m(\delta)_n 2^{2m+n} m!(n-2m)!}$$

Now substituting $n$ by $n+2m$ and making use of the result [7]

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\left[\frac{n}{2}\right]} C(m,n) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} C(m,n+2m),$$

we have

$$I = \frac{\Gamma(\gamma)\Gamma(\gamma)}{\Gamma(2\gamma)} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\alpha)_{n+2m}(\beta)_{n+2m} \ (\delta+1)_{n+2m}}{\left(\frac{1}{2}(\alpha+\beta+3)\right)_{n+2m} (\gamma+\frac{1}{2})_m(\delta)_{n+2m} 2^{n+4m} m!n!}$$

Using the identity

$$(\alpha)_{n+2m} = (\alpha)_{2m}(\alpha+2m)_n$$

and after some calculations, we have

$$\begin{aligned} I =& \frac{\Gamma(\gamma)\Gamma(\gamma)}{\Gamma(2\gamma)} \sum_{m=0}^{\infty} \frac{(\alpha)_{2m}(\beta)_{2m} \ (\delta+1)_{2m}}{\left(\frac{1}{2}(a+b+3)\right)_{2m} (\gamma+\frac{1}{2})_m(\delta)_{2m} 2^{4m} m!} \\ & \times \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n \ (\delta+1)_n}{\left(\frac{1}{2}(\alpha+\beta+3)+2m\right)_n (\delta+2m)_n 2^n n!} \end{aligned}$$

Summing up the inner series, we have

$$I = \frac{\Gamma(\gamma)\Gamma(\gamma)}{\Gamma(2\gamma)} \sum_{m=0}^{\infty} \frac{(\alpha)_{2m}(\beta)_{2m} \quad (\delta+1)_{2m}}{\left(\frac{1}{2}(\alpha+\beta+3)\right)_{2m} (\gamma+\frac{1}{2})_m (\delta)_{2m} 2^{4m} m!}$$
$$\times\, {}_3F_2 \left[ \begin{matrix} \alpha+2m, \beta+2m, \delta+1+2m \\ \frac{1}{2}(\alpha+\beta+3)+2m, \delta+2m \end{matrix} ; \frac{1}{2} \right]$$

We now observe that ${}_3F_2$ appearing here can be computed with the help of (1.1) and after much simplification, separating into two terms and finally applying (1.7), we get

$$I = \frac{\Gamma(\gamma)\Gamma(\gamma)}{\Gamma(2\gamma)} \frac{2^{\alpha+\beta-2}\Gamma\left(\gamma+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}(\alpha+\beta+3)\right)\Gamma\left(\gamma-\frac{1}{2}(\alpha+\beta+1)\right)}{(\alpha-\beta-1)(\alpha-\beta+1)\Gamma(\frac{1}{2})\Gamma(\alpha)\Gamma(\beta)}$$
$$\times \left\{ \gamma_1 \frac{\Gamma(\frac{1}{2}\alpha)\Gamma(\frac{1}{2}\beta)}{\Gamma\left(\gamma-\frac{1}{2}\alpha+\frac{1}{2}\right)\Gamma\left(\gamma-\frac{1}{2}\beta+\frac{1}{2}\right)} + \gamma_2 \frac{\Gamma\left(\frac{1}{2}(\alpha+1)\right)\Gamma\left(\frac{1}{2}(\beta+1)\right)}{\Gamma\left(\gamma-\frac{1}{2}\alpha\right)\Gamma\left(\gamma-\frac{1}{2}\beta\right)} \right\} \tag{2.3}$$

Finally, equating the results (2.2) and (2.3), we obtain the required result (1.3). The new proof of (1.3) is completed.

# 3 Application

In this section, by employing extended Watson summation theorem (1.3), we shall evaluate the following two integrals involving generalized hypergeometric function ${}_3F_2$. These are

$$\int_0^{\frac{\pi}{2}} e^{2i\gamma\theta}(sin\theta)^{\gamma-1}(cos\theta)^{\gamma-1}{}_3F_2 \left[ \begin{matrix} \alpha, \beta, \delta+1 \\ \frac{1}{2}(\alpha+\beta+3), \delta \end{matrix} ; e^{i\theta} cos\theta \right] d\theta$$
$$= e^{\frac{i\pi r}{2}} \frac{\Gamma(\gamma)\Gamma(\gamma)}{\Gamma(2\gamma)} \Omega \tag{3.1}$$

provided $Re(\gamma) > 0$ and $Re(2\gamma-\alpha-\beta) > 1$. Where $\Omega$ is the same as given in (1.3).

$$\int_0^{\frac{\pi}{2}} e^{2i\gamma\theta}(sin\theta)^{\gamma-1}(cos\theta)^{\gamma-1}{}_3F_2 \left[ \begin{matrix} \alpha, \beta, \delta+1 \\ \frac{1}{2}(\alpha+\beta+3), \delta \end{matrix} ; e^{i(\theta-\frac{\pi}{2})} sin\theta \right] d\theta$$
$$= e^{\frac{i\pi\gamma}{2}} \frac{\Gamma(\gamma)\Gamma(\gamma)}{\Gamma(2\gamma)} \Omega \tag{3.2}$$

provided $Re(\gamma) > 0$ and $Re(2\gamma-\alpha-\beta) > 1$. Where $\Omega$ is the same as given in (1.3).

*Proof.* To prove result (3.1), we proceed as follows.
Let us denote the left hand side of (3.1) by I, we have

$$I = \int_0^{\frac{\pi}{2}} e^{2i\gamma\theta}(sin\theta)^{\gamma-1}(cos\theta)^{\gamma-1}{}_3F_2 \left[ \begin{matrix} \alpha, \beta, \delta+1 \\ \frac{1}{2}(\alpha+\beta+3)_n, \delta \end{matrix} ; e^{i\theta} cos\theta \right] d\theta$$

Next, expressing generalized hypergeometric function ${}_3F_2$ involved in the process as a series, changing the order of integration and summation, which is easily seen to be justified due to the uniform convergence of the series involved in the process, we have

$$I = \sum_{m=0}^{\infty} \frac{(\alpha)_n(\beta)_n \quad (\delta+1)_n}{\left(\frac{1}{2}(\alpha+\beta+3)\right)_n (\delta)_n n!}$$
$$\times \int_0^{\frac{\pi}{2}} e^{i(2\gamma+n)\theta}(sin\theta)^{\gamma-1}(cos\theta)^{\gamma+n-1} d\theta$$

Now, evaluating the integral with the help of the known integral due to MacRobert [7], viz

$$\int_0^{\frac{\pi}{2}} e^{i(\alpha+\beta)\theta}(sin\theta)^{\alpha-1}(cos\theta)^{\beta-1}d\theta = e^{\frac{i\pi\alpha}{2}}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

provided $Re(\alpha) > 0$ and $Re(\beta) > 0$. We have after some algebra

$$I = \sum_{m=0}^{\infty} \frac{(\alpha)_n(\beta)_n \quad (\delta+1)_n}{\left(\frac{1}{2}(\alpha+\beta+3)\right)_n (\delta)_n 2^{4m} n!} \times \frac{\Gamma(\gamma)\Gamma(\gamma+n)}{\Gamma(2\gamma+n)} e^{\frac{i\pi\gamma}{2}}$$

Using the relation

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$$

and after some simplification, we have

$$I = e^{\frac{i\pi\gamma}{2}}\frac{\Gamma(\gamma)\Gamma(\gamma)}{\Gamma(2\gamma)} \sum_{m=0}^{\infty} \frac{(\alpha)_n(\beta)_n(\gamma)_n \quad (\delta+1)_n}{\left(\frac{1}{2}(\alpha+\beta+3)\right)_n (2\gamma)_n(\delta)_n n!}$$

Summing up the series, we get

$$I = e^{\frac{i\pi\gamma}{2}}\frac{\Gamma(\gamma)\Gamma(\gamma)}{\Gamma(2\gamma)} {}_4F_3\left[\begin{matrix} \alpha, \beta, \gamma, \delta+1 \\ \frac{1}{2}(\alpha+\beta+3), 2\gamma, \delta \end{matrix} ;1\right].$$

We now observe that the term appearing on right hand side can be evaluated with the help of the result (1.3), and we easily arrive at the right hand side of result (3.1). This completes the proof of result (3.1). □

In exactly the similar way, result (3.2) can be established. So we left this as an exercise to the interested readers.

**Remark**: For another proof of extended Watson theorem (1.3), we refer to a paper [4].

# 4 Special Case

It is seen that in both cases, by letting $\beta = -2n$ and replacing $\alpha$ by $\alpha + 2n$, or by letting $\beta = -2n - 1$ and replacing $\alpha$ by $\alpha + 2n + 1$, where $n$ is zero or a positive integer, one of the two terms appearing on the left hand side of integrals (3.1) and (3.2) will vanish and we can easily obtain four new and interesting results. But we shall not record here due to the lack of space.

# 5 Concluding Remark

In this work, we have given a new proof of extended Watson summation theorem. As an application, we have evaluated two new and interesting integrals involving a generalized hypergeometric function.

# References


[1] Bailey, W. N., 1928, Product of generalized hypergeometric series, *Proc. London Math. Soc.,* 28(2), 242-254.

[2] Bailey, W. N., 1935, Generalized hypergeometric series, *Cambridge University Press*, Cambridge Reprinted by Stechert-Hafner, New York (1964).

[3] Choi, J., Rohira, V., and Rathie, A.K., 2017, Note on extended Watson summation theorem for the series ${}_4F_3(1)$, *Far East Journal of Math. Sci.*, 102(11), 2895-2902.

[4] Jun, S., and Kilicman, A., 2018, A new proof of extended Watson summation theorem, *Appl. Math. Inf. Sci.*, 12(3), 633-636.

[5] Kim, Y. S., Rakha, M. A., and Rathie, A. K., 2010, Extension of certain classical summations for the series ${}_2F_1$, ${}_3F_2$ and ${}_4F_3$, with applications in Ramanujan's summations, *Int. J. of Mathematics and Mathematical Sciences*, Article ID 309503, 26 pages.

[6] Lavoie, J. L., Grondin, F., and Rathie, A. K., 1992, Generalization of Watson summation theorem on the a ${}_3F_2$, *Indian J. Math* , 34(2), 23-32.

[7] MacRobert, T. M., 1961, Beta function formulae and integrals involving E- functions, *Math Ann.*, 142, 450-452.

[8] Rainville, E. D., 1971, Special functions, *The Macmillan Company*, New York (1960), Reprinted by Chelsea Publishing Company, Bronx, New York (1971).