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New Methods for Order $n = 4k + 2$ via Vertical Self-Complementarity and a Framework for Finding All Even-Order Magic Squares

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Abstract: *The discovery of analytical methods for constructing magic squares of order $n = 8t - 2$ is extremely rare. In this article, we present such a discovery. The magic squares generated have vertical symmetry and are numerous within the same order. The only drawback is that we were unable to discover a method for orders of the type n=8t+2. Most of the few existing methods for these orders require auxiliary magic squares, unlike ours, which is a straightforward method and can be better used in all branches of technology. Furthermore, these magic squares possess exceptional symmetry properties and, from the construction of one of them, many others (at least $(n-2)!^{\frac{n}{2}}$) of the same order are immediately obtained due to their vertical complementarities ($E = (e_{u,v})_{u,v \in I_n}, e_{u,v} + e_{u,n+1-v} = n^2 + 1, \forall u, v \in I_n$). We also present analytical and algebraic algorithms for constructing a large number of magic squares for all remaining even orders. The abundance of magic squares constructed by the algorithms led us to establish a project for the construction of all magic squares of all even orders. We have established a mathematical theory of the joint magicization of the diagonals of semimagic squares of even orders.*

Keywords: Magic squares, Analytical methods, Complementarity, Diagonalization, Abundance.

1 Introduction

There are many methods for constructing magic squares for which there are still no known proofs that for all orders in which they are applied, they actually generate magic squares. The results that we present here, for some orders of the type $n = 4k + 2, k \in \mathrm{N}^*$, not only are they demonstrated, but we also establish matrix formulas for them; that is, we write each entry of the magic square as a quadratic function of its order n and a linear function the order (i, j) of the 2×2 block matrix that forms it. Here, we will follow the notations of (Danielsson, Holger. (2024)). This author says:

> "Magic squares of single-even order are much more difficult to construct than magic squares of double-even order. Because for the latter the half-order is again an even integer, symmetric arrangements of the numbers can be easily exploited. This is not possible with single-even orders because the half order is an odd number here. Therefore, additional exchanges are always necessary for these orders" (op. cit. p. 380).

The last major comprehensive discovery on methods of this order was the famous Medjig method (op. cit. p. 551) in 2006, but unfortunately, even this method uses magic squares of lower orders (such as the Lo Shu for the 6th order) as aids in its construction. In (de Oliveira Miranda (2020a), p. 34) we had already noted that the magic squares which we call the magic squares of the Lohans (Arhats), $M = (m_{u,v})_{u,v \in I_n}$, are horizontally self-complementary, that is, $m_{u,v} + m_{u,n+1-v} = n^2 + 1, \forall u, v \in I_n$. With regard to vertical self-complementarity, this study follows the Himalayan tradition to which we refer in (ibid., p. 31). Analytical methods for constructing magic squares of orders of the type $n = 8t - 2, t \in \mathrm{N}^*$, usually, require other methods for constructing magic squares to create auxiliary magic squares beforehand, such as Strachey's method and Conway's LUX method. The great advantage of the method presented and proven here is that it does not require other methods to create its infinite magic squares, for all orders of the type $n = 8t - 2$. Furthermore, these magic squares possess exceptional symmetry properties and, from the construction of one of them, many others (at least $(n-2)!^{\frac{n}{2}}$) of the same order are immediately obtained due to their vertical complementarities. In (Manjunath, H. (2020)), graph theory and Conway's ideas have been used to study some simply even-order magic squares. In (Chong, J. Y. (2025)) the same types of searches are performed. In (Miranda (2021), pp. 19-21)), we did not study magic squares of simply even orders, but some ideas consolidated here were initially presented there, as well as in (de Oliveira Miranda (2020b), pp. 13-14), especially the large number of magic squares found within the same order. In (Rani, N., 2023) and (Rani, N., 2025) very comprehensive applications of magic squares are made. Here, we present a method for orders of the type $n = 8t - 2$, and this method generates a very large number of other magic squares within the same order n. We did not obtain a method for orders of the type $n = 8t + 2$, but our search generated techniques so powerful that it led us to launch this bold project.

2 Basic Concepts and Notations

A magic square of order $n \in \mathrm{N}^* - \{2\}$ is a square matrix formed by all the numbers $1, 2, 3, ..., n^2$ and such that the sum of the numbers in each row, each column and each of the two diagonals is equal to $c_n = \frac{n^3+n}{2}$. We call c_n the magic constant. A magic square of order n is non-normal when the sums of the numbers in the rows, columns and diagonals are all equal, but not equal to c_n or the set of numbers that form it is not $I_{n^2} = \{1, 2, 3, ..., n^2\}$. We call these sums totals. When we remove the assumptions about the diagonals, we call them semimagic squares. In mathematical, logical, and computational terms, swapping one thing for another (a permutation or swap operation) means altering the position or value of two elements within a system, so that each one comes to occupy the place that previously belonged to the other.

3 New method for constructing magic squares of order $n = 8k - 2$

Proposition 1. Let $I_n = \{1, 2, 3, ..., n\}$, n a positive even natural number greater than 2. Consider the matrix $A = (a_{u,v})_{u,v \in I_n} = (A_{i,j})_{i,j \in I_{\frac{n}{2}}}$ of even order formed by blocks of order 2 as follows:

$$A_{i,j} = \begin{pmatrix} (A_{i,j})_{1,1} & (A_{i,j})_{1,2} \\ (A_{i,j})_{2,1} & (A_{i,j})_{2,2} \end{pmatrix} = \begin{pmatrix} -1 + 2j + 2ni - 2n & n^2 + 1 - 2ni + 2n - 2j \\ 2ni - 2j + 2 & n^2 - 2ni + 2j \end{pmatrix} \quad (1)$$

Then:

$$(A_{i,j})_{1,1} + \left(A_{\frac{n}{2}+1-i,j}\right)_{2,1} = n^2 + 1; (A_{i,j})_{1,2} + \left(A_{\frac{n}{2}+1-i,j}\right)_{2,2} = n^2 + 1; (A_{i,j})_{2,1} + \left(A_{\frac{n}{2}+1-i,j}\right)_{1,1} = n^2 + 1; (A_{i,j})_{2,2} + \left(A_{\frac{n}{2}+1-i,j}\right)_{1,2} = n^2 + 1, \forall i, j \in I_{\frac{n}{2}} \quad (2)$$

$$a_{u,v} + a_{n+1-u,v} = a_{u,\tilde{v}} + a_{n+1-u,\tilde{v}} = n^2 + 1, \forall u, v, \tilde{v} \in I_n \quad (3)$$

$$\sum_{i=1}^{\frac{n}{2}} \left((A_{i,j})_{1,1} + (A_{i,j})_{2,1}\right) = \sum_{i=1}^{\frac{n}{2}} \left((A_{i,j})_{1,2} + (A_{i,j})_{2,2}\right) = c_n, \forall j \in I_{\frac{n}{2}} \quad (4)$$

$$\sum_{j=1}^{\frac{n}{2}} \left((A_{i,j})_{1,1} + (A_{i,j})_{1,2}\right) = c_n - \frac{n}{2}, \forall i \in I_{\frac{n}{2}} \quad (5)$$

$$\sum_{j=1}^{\frac{n}{2}} \left((A_{i,j})_{2,1} + (A_{i,j})_{2,2}\right) = c_n + \frac{n}{2}, \forall i \in I_{\frac{n}{2}} \quad (6)$$

$$\sum_{i=1}^{\frac{n}{2}} \left((A_{i,i})_{1,1} + (A_{i,i})_{2,2}\right) = c_n - \frac{n^2}{2} \quad (7)$$

$$\sum_{i=1}^{\frac{n}{2}} \left(\left(A_{i,\frac{n}{2}+1-i}\right)_{1,2} + \left(A_{i,\frac{n}{2}+1-i}\right)_{2,1}\right) = c_n + \frac{n^2}{2}. \quad (8)$$

Demonstration.
(2). $(A_{i,j})_{1,1} + \left(A_{\frac{n}{2}+1-i,j}\right)_{2,1} = -1 + 2j + 2ni - 2n + 2n\left(\frac{n}{2} + 1 - i\right) - 2j + 2 = n^2 + 1$. Similarly, the other three identities can be proven through direct inspection.
(3). This is a direct consequence of (2).
(4). $\sum_{i=1}^{\frac{n}{2}} \left((A_{i,j})_{1,1} + (A_{i,j})_{2,1}\right) = \sum_{i=1}^{\frac{n}{2}} ((-1 + 2j + 2ni - 2n) + (2ni - 2j + 2)) = \sum_{i=1}^{\frac{n}{2}} (1 + 4ni - 2n) = \frac{n^3+n}{2} = c_n$. $\sum_{i=1}^{\frac{n}{2}} \left((A_{i,j})_{1,2} + (A_{i,j})_{2,2}\right) = \sum_{i=1}^{\frac{n}{2}} \left((n^2 + 1 - 2ni + 2n - 2j) + (n^2 - 2ni + 2j)\right) =$

$\sum_{i=1}^{\frac{n}{2}} \left(2n^2+1-4ni+2n\right) = c_n$. Similarly, by direct inspection, items (5), (6), (7) and (8) are proven.
Corollary 1. If a square matrix satisfies hypotheses (2) or (3) then we can swap $a_{u,v}$ with $a_{u,\tilde{v}}$ and $a_{n+1-u,v}$ with $a_{n+1-u,\tilde{v}}$ and, even so, the sums of the elements in the rows and columns of the matrix do not change.

Corollary 2. Under the assumptions of Proposition 1, it holds true

$$(A_{i,j})_{1,1} \equiv (A_{i,\frac{n}{2}+1-j})_{1,2} \pmod n,\ (A_{i,j})_{1,2} \equiv (A_{i,\frac{n}{2}+1-j})_{1,1} \pmod n,\ (A_{i,j})_{2,1} \equiv (A_{i,\frac{n}{2}+1-j})_{2,2} \pmod n,\ (A_{i,j})_{2,2} \equiv (A_{i,\frac{n}{2}+1-j})_{2,1} \pmod n, \tag{9}$$

due to identities

$$(A_{i,j})_{1,1} - (A_{i,\frac{n}{2}+1-j})_{1,2} = n(4i-3-n),\ (A_{i,j})_{1,2} - (A_{i,\frac{n}{2}+1-j})_{1,1} = n(4i-3-n),$$
$$(A_{i,j})_{2,1} - (A_{i,\frac{n}{2}+1-j})_{2,2} = n(4i-3-n) \text{ and } (A_{i,j})_{2,2} - (A_{i,\frac{n}{2}+1-j})_{2,1} = n(4i-3-n) \tag{10}$$

Proposition 2. If in any column of order j, $j \in I_{\frac{n}{2}}$, of matrix A we swap $(A_{i,j})_{1,1}$ with $(A_{i,j})_{2,1}$, $\forall i \in I_{\frac{n}{2}}$, (alternatively, we swap $(A_{i,j})_{1,2}$ with $(A_{i,j})_{2,2}$, $\forall i \in I_{\frac{n}{2}}$), then the resulting matrix $R = (R_{i,j})_{i,j} = (r_{u,v})_{u,v\in I_n}$ will be such that $r_{n+1-u,v} + r_{u,v} = n^2+1$.

Demonstration. For the remaining columns, matrices A and R coincide. Therefore, we only need to inspect the column of order j. From the construction of R, we have that $(R_{i,j})_{1,1} = (A_{i,j})_{2,1}$, $(R_{i,j})_{1,2} = (A_{i,j})_{2,2}$, $(R_{i,j})_{2,1} = (A_{i,j})_{1,1}$, $(R_{i,j})_{2,2} = (A_{i,j})_{1,2}$. From these four equalities and from (2) of the Proposition 1, the result follows.

Proposition 3. Let A, be defined in (1), of order $n = 8t-2, t \in \mathbb{N}^*$. If in A we swap $(A_{i,3t})_{1,1}$ with $(A_{i,3t})_{2,1}, \forall i \in I_{\frac{n}{2}}$, then we will obtain a semimagic square B. The sum of the elements of the main diagonal of B will be equal to $c_n - \frac{n^2}{2} + \frac{n}{2}$ and the sum of the elements of the secondary diagonal of B will be equal to $c_n + \frac{n^2}{2} - \frac{n}{2}$.

Demonstration. $(A_{i,3t})_{2,1} - (A_{i,3t})_{1,1} = (2ni+2-6t) - (-1+6t+2ni-2n) = \frac{n}{2}, \forall i \in I_{\frac{n}{2}}$. From items (4), (5), (6) of Proposition 1 and this equality it follows that after these $\frac{n}{2}$ swaps we obtain a normal semimagic square B with a magic constant equal to c_n. From $\left(A_{\frac{n}{2}+1-3t,3t}\right)_{2,1} - \left(A_{\frac{n}{2}+1-3t,3t}\right)_{1,1} = -(-1+6t+2n(\frac{n}{2}+1-3t)-2n) + (2n(\frac{n}{2}+1-3t)-6t+2) = \frac{n}{2}$ we conclude that the secondary diagonal loses $\frac{n}{2}$ units and, from $(A_{3t,3t})_{2,1} - (A_{3t,3t})_{1,1}$ we conclude that the main diagonal gains $\frac{n}{2}$ units.

Proposition 4. If in matrix $B = (B_{i,j})_{i,j\in I_{\frac{n}{2}}}$ of Proposition 3 we swap $(B_{2t,2t})_{1,1} = \frac{n^2}{2} - \frac{n}{2}$ with $(B_{2t,2t})_{1,2} = \frac{n^2}{2} + \frac{n}{2}$ and swap $(B_{2t,2t})_{2,1} = \frac{n^2}{2} + \frac{n}{2} + 1$ with $(B_{2t,2t})_{2,2} = \frac{n^2}{2} - \frac{n}{2} + 1$ we obtain a semimagic square C. The sum of the elements of the main diagonal of C is equal to $c_n - \frac{n^2}{2} + \frac{n}{2} + 2n$ and the sum of the elements of the secondary diagonal of C is $c_n + \frac{n^2}{2} - \frac{n}{2} - 2n$.

Demonstration. These values for $(B_{2t,2t})_{1,1}$, $(B_{2t,2t})_{1,2}$, $(B_{2t,2t})_{2,1}$ and $(B_{2t,2t})_{2,2}$ can be calculated from (1), of Proposition 1, since matrices A and B differ only in the $j = 3t$ column. Given that $(B_{2t,2t})_{1,1} + (B_{2t,2t})_{2,1} = n^2+1$ and $(B_{2t,2t})_{1,2} + (B_{2t,2t})_{2,2} = n^2+1$, the swaps made do not alter the sums of the elements in the rows or columns. On the other hand, $(B_{2t,2t})_{1,2}$-$(B_{2t,2t})_{1,1} = n$ indicates that the main diagonal gains n units and the secondary diagonal loses n units and, $(B_{2t,2t})_{1,2} - (B_{2t,2t})_{1,1} = n$ also indicates that the main diagonal gains n units and the secondary diagonal loses n units.

Proposition 5. If in matrix $C = (C_{i,j})_{i,j\in I_{\frac{n}{2}}}$ of Proposition 4 we swap $(C_{2t,2t})_{1,1}$ with $(C_{2t,3t})_{1,1}$ and swap $(C_{2t,2t})_{2,1}$ with $(C_{2t,3t})_{2,1}$ we obtain a semimagic square D. The sum of the elements of the main diagonal of D is equal to $c_n - \frac{n^2}{2} + \frac{n}{2} + 2n - (\frac{n}{4} - \frac{1}{2})$ and the sum of the elements of the secondary diagonal of C is $c_n + \frac{n^2}{2} - \frac{n}{2} - 2n + (\frac{n}{4} - \frac{1}{2})$.

Demonstration. From propositions 3 and 4 (aided by proposition 2) we have:
$(A_{2t,2t})_{1,2} = (B_{2t,2t})_{1,2} = (C_{2t,2t})_{1,1}$, $(C_{2t,3t})_{1,1} = (B_{2t,3t})_{1,1} = (A_{2t,3t})_{2,1}$; $(A_{2t,2t})_{2,2} = (B_{2t,2t})_{2,2} = (C_{2t,2t})_{2,1}$, $(C_{2t,3t})_{2,1} = (B_{2t,3t})_{2,1} = (A_{2t,3t})_{1,1}$. Therefore, from these equalities and (1) it follows: $(C_{2t,2t})_{1,1} - (C_{2t,3t})_{1,1} = (n^2+1-4nt+2n-4t) - (4nt-6t+2) = \frac{n}{4} - \frac{1}{2}$; $(C_{2t,2t})_{2,1} - (C_{2t,3t})_{2,1} = (n^2-4nt+4t) - (-1+6t+4nt-2n) = -8nt-2t+n^2+2n+1 = -(\frac{n}{4} - \frac{1}{2})$. Since $(C_{2t,2t})_{1,1}$ is on the main

diagonal and $(C_{2t,2t})_{2,1}$ is on the secondary diagonal, the main diagonal loses $\frac{n}{4} - \frac{1}{2}$ units and the secondary diagonal gains $\frac{n}{4} - \frac{1}{2}$ units. Note that for $n = 8t - 2$, $(C_{2t,2t})_{1,1} + (C_{2t,2t})_{2,1} = (C_{2t,3t})_{1,1} + (C_{2t,3t})_{2,1}$, indicating that simultaneous swaps do not alter the sums of the elements in the rows or columns.

Theorem 1. In the semimagic square D, mentioned in Proposition 5, if we swap $(D_{t+1,t+1})_{1,1}$ with $(D_{t+1,\frac{n}{2}})_{1,2}$ and also swap $(D_{\frac{n}{2}+1-(t+1),t+1})_{2,1}$ with $(D_{\frac{n}{2}+1-(t+1)})_{2,2}$, we will obtain a normal magic square E of order $n = 8t - 2$.

Demonstration. D and A coincide in rows $i = t + 1$ and $i = \frac{n}{2} + 1 - (t + 1)$ and in columns $j = t + 1$ and $j = \frac{n}{2}$. Therefore, the increment in the main diagonal will be equal to $(D_{t+1,\frac{n}{2}})_{1,2} - (D_{t+1,t+1})_{1,1} = (n^2 + 1 - 2n(t + 1) + 2n - 2\frac{n}{2}) - (-1 + 2(t + 1) + 2n(t + 1) - 2n) = n^2 - 4nt - n - 2t$. The sum of the elements on the main diagonal of E will be equal to this value plus the sum of the elements on the main diagonal of D, that is: $(c_n - \frac{n^2}{2} + \frac{n}{2} + 2n - (\frac{n}{4} - \frac{1}{2})) + (n^2 - 4nt - n - 2t) = c_n$. The increment in the secondary diagonal is equal to

$$(D_{\frac{n}{2}-t,\frac{n}{2}})_{2,2} - (D_{\frac{n}{2}-t,t+1})_{2,1} = (n^2 - 2n(\tfrac{n}{2} - t) + 2\tfrac{n}{2}) - (2n(\tfrac{n}{2} - t) - 2(t + 1) + 2) = 4nt + n - n^2 + 2t. \quad (11)$$

The sum of the elements on the secondary diagonal of E will be equal to this value plus the sum of the elements on the secondary diagonal of D, that is: $c_n + \frac{n^2}{2} - \frac{5n}{2} + \frac{n}{4} - \frac{1}{2} + (4nt + n - n^2 + 2t) = c_n$.

Example 1 ($t = 1$; $n = 6$; $j(6) = 3$; $c_6 = 111$; $i(1) = 2$)

$$A = \begin{pmatrix} 1 & 35 & 3 & 33 & 5 & 31 \\ 12 & 26 & 10 & 28 & 8 & 30 \\ 13 & 23 & 15 & 21 & 17 & 19 \\ 24 & 14 & 22 & 16 & 20 & 18 \\ 25 & 11 & 27 & 9 & 29 & 7 \\ 36 & 2 & 34 & 4 & 32 & 6 \end{pmatrix}; B = \begin{pmatrix} 1 & 35 & 3 & 33 & \mathbf{8} & 31 \\ 12 & 26 & 10 & 28 & \mathbf{5} & 30 \\ 13 & 23 & 15 & 21 & \mathbf{20} & 19 \\ 24 & 14 & 22 & 16 & \mathbf{17} & 18 \\ 25 & 11 & 27 & 9 & \mathbf{32} & 7 \\ 36 & 2 & 34 & 4 & \mathbf{29} & 6 \end{pmatrix};$$

$$C = \begin{pmatrix} 1 & 35 & 3 & 33 & 8 & 31 \\ 12 & 26 & 10 & 28 & 5 & 30 \\ 13 & 23 & \mathbf{21} & \mathbf{15} & 20 & 19 \\ 24 & 14 & \mathbf{16} & \mathbf{22} & 17 & 18 \\ 25 & 11 & 27 & 9 & 32 & 7 \\ 36 & 2 & 34 & 4 & 29 & 6 \end{pmatrix}; D = \begin{pmatrix} 1 & 35 & 3 & 33 & 8 & 31 \\ 12 & 26 & 10 & 28 & 5 & 30 \\ 13 & 23 & \mathbf{20} & 15 & \mathbf{21} & 19 \\ 24 & 14 & \mathbf{17} & 22 & \mathbf{16} & 18 \\ 25 & 11 & 27 & 9 & 32 & 7 \\ 36 & 2 & 34 & 4 & 29 & 6 \end{pmatrix},$$

$$E = \begin{pmatrix} 1 & 35 & 3 & 33 & 8 & 31 \\ 12 & \mathbf{30} & 10 & 28 & 5 & \mathbf{26} \\ 13 & 23 & 20 & 15 & 21 & 19 \\ 24 & 14 & 17 & 22 & 16 & 18 \\ 25 & \mathbf{7} & 27 & 9 & 32 & \mathbf{11} \\ 36 & 2 & 34 & 4 & 29 & 6 \end{pmatrix} \quad (12)$$

Note that according to the fundamental principle of counting, we can make more $(n-2)!^{\frac{n}{2}} - 1$ other magic squares without changing the position of the numbers that are already on the diagonals. In this Example 1 we can use Corollary 1 to find more $(6-2)!^{\frac{6}{2}} - 1$ additional magic squares.

Comment 1. The essence of the proof consists of making successive swaps of numbers that are on the same lines using Corollary 1.

Construction of A: With the numbers from the set $X = \{1, 2, 3, ..., n^2\}$, $n = 8t - 2$ (t a positive natural number), construct the matrix A of order n. Construction of B: By constructing a matrix, there will be a bijective function between X and the set of entries of A;

Construction of B: In the $j = 3t$ column of A, swap the entry of the first row with the entry of the second row, the entry of the third row with the entry of the fourth row, the entry of the fifth row with the entry of the sixth row, and so on, until swapping the entry of the penultimate row with the entry of the last row. We will have formed matrix B, which is a semimagic square;

Construction of C: In the central 2×2 block

$$\begin{pmatrix} (B_{2t,2t})_{1,1} & (B_{2t,2t})_{1,2} \\ (B_{2t,2t})_{2,1} & (B_{2t,2t})_{2,2} \end{pmatrix}, \tag{13}$$

of B, $(B_{2t,2t})_{1,1}$ and $(B_{2t,2t})_{2,2}$ are on the main diagonal and $(B_{2t,2t})_{1,2}$ and $(B_{2t,2t})_{2,1}$ are on the secondary diagonal. Swapping $(B_{2t,2t})_{1,1}$ with $(B_{2t,2t})_{1,2}$ and $(B_{2t,2t})_{2,1}$ with $(B_{2t,2t})_{2,2}$ we will have made the semimagic square C;
Construction of D: If in C we swap, along the corresponding rows, the two elements of the first column of the central 2×2 block with the respective elements of the $j = 3t$ column, we will have formed the semimagic square D;

Construction of the magic square E: If in the first row of the double order i=t+1 of D we swap the number on the main diagonal with the corresponding number in the last column, and we also do the same in the complementary row of order i=(n/2)+1-(t+1), we will form a magic square E.

4 Complementarity

This study of vertical self-complementarity is very incipient and is part of the design spirit of this article. Our goal is to generalize the good properties of the matrix in (1) to try to find the largest possible number of magic squares of even orders. It was seen above that the matrix defined in (1) is very rich in very useful properties. These deserve a separate study, which will be done within our context, without losing our main focus, which is the construction of magic squares from the matrix defined in (1). As we define the objects, we will also add important information for the construction of magic squares.

Definition 1. If in any double block $A_{i,j}$ $(i, j \in I_{\frac{n}{2}})$ we swap $(A_{i,j})_{1,1}$ with $(A_{i,j})_{2,1}$ and $(A_{i,j})_{1,2}$ with $(A_{i,j})_{2,2}$ two units will be transferred from the bottom row to the top row and, $4n + 4 - 8j$ units will be transferred to the main diagonal of $A_{i,j}$. And the same quantity will be removed from the secondary diagonal of $A_{i,j}$. We will call this procedure an order reversal, or simply a reversal, if there is no ambiguity.

Definition 2. The reversals, being made in each of the $\frac{1}{2} + \frac{n}{4}$ double columns, entirely, and also the last single column entirely, cause the semimagic square B (Do not confuse it with matrix B from the previous section) to be such that $b_{u,v} + bn + 1 - u, v$ don't depend on v. We will say that B possesses the vertical self-complementary property or has complementarity. An immediate consequence of this property is that we can jointly swap, $b_{u,u}$ with $b_{u,v}$ and $b_{n+1-u,u}$ with $b_{n+1-u,v}$ $(\forall u, v \in I_n)$, by altering the sums of the elements on the two diagonals, together in opposite quantities, while maintaining the resulting matrix, it remains a semimagic square.

Definition 3. We can also define vertical self-complementarity as follows: a matrix $A = (a_{u,v})_{u,v \in I_n}$ has vertical self-complementary property if $a_{u,v} + a_{n+1-u,v} = a_{u,\tilde{v}} + a_{n+1-u,\tilde{v}}, \forall u, v, \tilde{v} \in I_n$.

Definition 4. The matrix A will have a vertical transversal self-complementarity property when the matrices of the broken diagonals have the vertical complementary property. The respective horizontal complementary properties are defined identically.

Comment 2. Note that if a magic square has the vertical self-complementary property, its 90^o rotation will have the horizontal self-complementary property. There is no matrix of order n with an entry set containing I_{n^2} which has complementary properties both vertically and horizontally simultaneously. This is a consequence of the following proposition.

Proposition 6. Let A a matrix of order n. If A has both vertical and horizontal complementary properties, then A is a centrosymmetric matrix.
Proof. Let $a_{u,v}$ any entry of A. Hypothetically, we have: $a_{u,v} + a_{u,n+1-v} = a_{n+1-u,v} + a_{n+1-u,n+1-v} = a_{u,v} + a_{n+1-u,v} = a_{u,n+1-v} + a_{n+1-u,n+1-v}$, and so it goes $a_{u,v} = a_{n+1-u,n+1-v}$ and $a_{u,n+1-v} = a_{n+1-u,v}$.

Comment 3. Note that in a matrix with vertical complementarity, the sum of the elements in any column is equal to the magic constant. If A is the matrix defined in (1), and if we swap $a_{u,v}$ with $a_{\tilde{u},\tilde{v}}$ and $a_{n+1-u,v}$ with $a_{n+1-\tilde{u},\tilde{v}}$, the resulting matrix will have a vertical auto-complementary property. Using the counting principle and a planar orientation argument, it can be seen that the number of matrices with vertical self-complementary property with the entry set containing I_{n^2} is $2^{\frac{n^2}{2}}\left(\frac{n^2}{2}\right)!$

5 Abundance of magic squares of order $n = 8t + 2, t \in \mathbb{N}^*$

For many centuries it has been known that it is very difficult to find methods to construct magic squares of simply even orders (Cf. Danielsson, H. (2022), pp. 469-494 or Danielsson, Holger. (2024), pp. 469-494). However, statistical results show that magic squares of these orders are not excessively scarce (Cf. Walter Trump. (2005)).

If we reverse, in the blocks of order 2 of matrix $A = (a_{u,v})_{u,v\in I_n} = (A_{i,j})_{i,j\in I_{\frac{n}{2}}}$, the top rows with the bottom rows, only to $i \in I_{\frac{n}{2}}, j \in I_{\frac{n}{4}+\frac{1}{2}}$ and swap $\left(A_{i,\frac{n}{2}}\right)_{2,2}$ with $\left(A_{i,\frac{n}{2}}\right)_{1,2}$ $(i \in I_{\frac{n}{2}})$, from the last column of A, we will have made a normal semimagic square $B = (b_{u,v})_{u,v\in I_n}$ which has $c_n + \left(\frac{n^2}{4} + n\right)$ units on the main diagonal and $c_n - \left(\frac{n^2}{4} + n\right)$ units on the secondary diagonal. To observe this, simply note that $\sum_{j=1}^{\frac{n}{4}+\frac{1}{2}} (4n + 4 - 8j) = \frac{3n^2}{4} + n - 1$, which, together with the unit in the last column, is the number of units that must be added to $SDP(A) = c_n - \frac{n^2}{2}$ units of the main diagonal. That is, the sum of the elements of the diagonal of is $SDP(B) = c_n + \left(\frac{n^2}{4} + n\right)$. To transform B into a magic square, we will make all the element swaps of the type $b_{u,u}$ with elements of the type $b_{u,v}$ such that any sums of differences $b_{u,v} - b_{u,u}$ be equal to $-\left(\frac{n^2}{4} + n\right)$. In more technical terms, we will seek to find numbers $\omega_{(u,v)} \in \{0, 1\}, u, v \in I_n$, such that

$$\sum_{v=1}^{n}\left(\sum_{u=1}^{n}\omega_{(u,v)}\left(b_{u,v} - b_{u,u}\right)\right) = -\left(\frac{n^2}{4} + n\right) \tag{14}$$

Here, if $\omega_{(u,v)} = \omega_{(u,v')} = 1$, then $v = v'$, so that there is no more than one swap with the same number on the main diagonal. If $\omega_{(u,v)} = 1$ the swap of $b_{u,u}$ with $b_{u,v}$ it is done. If $\omega_{(u,v)} = 0$ the swap of $b_{u,u}$ with $b_{u,v}$ it is not made. It's a different "equation", which can be solved by inspecting all the $2^{n^2} - n$ interesting possibilities (swaps within the main diagonal itself are not interesting, as they cancel each other out). If we assume that the $n^2 - n$ variables $\omega_{(u,v)}$ traverse the closed interval $[0, 1] \subset \mathbb{R}$, then the "matrix-equation" represents a Riemannian variety with a contained border in $\mathbb{R}^{n^2-n+1}$ and of dimension $n^2 - n$. When we fix values of n we obtain hyperplanes and when we fix the aforementioned variables we obtain parabolas. The magic squares resulting from (14) correspond to some points on the boundary of this Riemannian manifold of parabolic and also linear nature. Next, we write (14) using the coordinates $(i, j) \in I_{\frac{n}{2}} \times I_{\frac{n}{2}}$, of the double blocks

$$\begin{gathered}
-\left(\frac{n^2}{4} + n\right) = \sum_{i=1}^{\frac{n}{4}+\frac{1}{2}}\left(\sum_{j=1}^{\frac{n}{4}+\frac{1}{2}}\alpha_{(i,j)}((2ni - 2j + 2) - (2ni - 2i + 2))\right) + \\
\sum_{i=1}^{\frac{n}{4}+\frac{1}{2}}\left(\sum_{j=1}^{\frac{n}{4}+\frac{1}{2}}\beta_{(i,j)}((n^2 - 2ni + 2j) - (2ni - 2i + 2))\right) + \\
\sum_{i=1}^{\frac{n}{4}+\frac{1}{2}}\left(\sum_{j=1}^{\frac{n}{4}+\frac{1}{2}}\gamma_{(i,j)}((-1 + 2j + 2ni - 2n) - (n^2 + 1 - 2ni + 2n - 2i))\right) + \\
\sum_{i=1}^{\frac{n}{4}+\frac{1}{2}}\left(\sum_{j=1}^{\frac{n}{4}+\frac{1}{2}}\delta_{(i,j)}((n^2 + 1 - 2ni + 2n - 2j) - (n^2 + 1 - 2ni + 2n - 2j))\right) + \\
\sum_{i=1}^{\frac{n}{4}+\frac{1}{2}}\left(\sum_{j=\frac{n}{4}+\frac{3}{2}}^{\frac{n}{2}-1}\alpha_{(i,j)}((-1 + 2j + 2ni - 2n) - (2ni - 2i + 2))\right) + \\
\sum_{i=1}^{\frac{n}{4}+\frac{1}{2}}\left(\sum_{j=\frac{n}{4}+\frac{3}{2}}^{\frac{n}{2}-1}\beta_{(i,j)}((n^2 + 1 - 2ni + 2n - 2j) - (2ni - 2i + 2))\right) + \\
\sum_{i=1}^{\frac{n}{4}+\frac{1}{2}}\left(\sum_{j=\frac{n}{4}+\frac{3}{2}}^{\frac{n}{2}-1}\gamma_{(i,j)}((2ni - 2j + 2) - (n^2 + 1 - 2ni + 2n - 2i))\right) + \\
\sum_{i=1}^{\frac{n}{4}+\frac{1}{2}}\left(\sum_{j=\frac{n}{4}+\frac{3}{2}}^{\frac{n}{2}-1}\delta_{(i,j)}((n^2 - 2ni + 2j) - (n^2 + 1 - 2ni + 2n - 2i))\right) + \\
\sum_{i=\frac{n}{4}+\frac{3}{2}}^{\frac{n}{2}-1}\left(\sum_{j=1}^{\frac{n}{4}+\frac{1}{2}}\alpha_{(i,j)}((2ni - 2j + 2) - (-1 + 2i + 2ni - 2n))\right) + \\
\sum_{i=\frac{n}{4}+\frac{3}{2}}^{\frac{n}{2}-1}\left(\sum_{j=1}^{\frac{n}{4}+\frac{1}{2}}\beta_{(i,j)}((n^2 - 2ni + 2j) - (-1 + 2i + 2ni - 2n))\right) + \\
\sum_{i=\frac{n}{4}+\frac{3}{2}}^{\frac{n}{2}-1}\left(\sum_{j=1}^{\frac{n}{4}+\frac{1}{2}}\gamma_{(i,j)}((-1 + 2j + 2ni - 2n) - (n^2 - 2ni + 2i))\right) +
\end{gathered}$$

$$\sum_{i=\frac{n}{4}+\frac{3}{2}}^{\frac{n}{2}-1}\left(\sum_{j=1}^{\frac{n}{4}+\frac{1}{2}}\delta_{(i,j)}((n^2+1-2ni+2n-2j)-(n^2-2ni+2i))\right)+$$
$$\sum_{i=\frac{n}{4}+\frac{3}{2}}^{\frac{n}{2}-1}\left(\sum_{j=\frac{n}{4}+\frac{3}{2}}^{\frac{n}{2}-1}\alpha_{(i,j)}((-1+2j+2ni-2n)-(-1+2i+2ni-2n))\right)+$$
$$\sum_{i=\frac{n}{4}+\frac{3}{2}}^{\frac{n}{2}-1}\left(\sum_{j=\frac{n}{4}+\frac{3}{2}}^{\frac{n}{2}-1}\beta_{(i,j)}((n^2+1-2ni+2n-2j)-(-1+2i+2ni-2n))\right)+$$
$$\sum_{i=\frac{n}{4}+\frac{3}{2}}^{\frac{n}{2}-1}\left(\sum_{j=\frac{n}{4}+\frac{3}{2}}^{\frac{n}{2}-1}\gamma_{(i,j)}((2ni-2j+2)-(n^2-2ni+2i))\right)+$$
$$\sum_{i=\frac{n}{4}+\frac{3}{2}}^{\frac{n}{2}-1}\left(\sum_{j=\frac{n}{4}+\frac{3}{2}}^{\frac{n}{2}-1}\delta_{(i,j)}((n^2-2ni+2j)-(n^2-2ni+2i))\right)+\sum_{j=1}^{\frac{n}{4}+\frac{1}{2}}\alpha_{(\frac{n}{2},j)}((n^2-2j+2)-(-1-n+n^2))+\sum_{j=1}^{\frac{n}{4}+\frac{1}{2}}\beta_{(\frac{n}{2},j)}((2j)-(-1-n+n^2))+\sum_{j=1}^{\frac{n}{4}+\frac{1}{2}}\gamma_{(\frac{n}{2},j)}((-1+2j+n^2-2n)-(n+1))+\sum_{j=1}^{\frac{n}{4}+\frac{1}{2}}\delta_{(\frac{n}{2},j)}((1+2n-2j)-(n+1))+\sum_{j=\frac{n}{4}+\frac{3}{2}}^{\frac{n}{2}-1}\alpha_{(\frac{n}{2},j)}((-1+2j+n^2-2n)-(-1-n+n^2))+\sum_{j=\frac{n}{4}+\frac{3}{2}}^{\frac{n}{2}-1}\beta_{(\frac{n}{2},j)}((1+2n-2j)-(-1-n+n^2))+\sum_{j=\frac{n}{4}+\frac{3}{2}}^{\frac{n}{2}-1}\gamma_{(\frac{n}{2},j)}((n^2-2j+2)-(n+1))+\sum_{j=\frac{n}{4}+\frac{3}{2}}^{\frac{n}{2}-1}\delta_{(\frac{n}{2},j)}((2j)-(n+1))+\beta_{(\frac{n}{2},\frac{n}{2})}\left(n-\left(-1-n+n^2\right)\right)+\gamma_{(\frac{n}{2},\frac{n}{2})}\left((n^2-n+2)-(n+1)\right)+\sum_{i=1}^{\frac{n}{4}+\frac{1}{2}}\alpha_{(i,\frac{n}{2})}((-1-n+2ni)-(2ni-2i+2))+\sum_{i=1}^{\frac{n}{4}+\frac{1}{2}}\gamma_{(i,\frac{n}{2})}((2ni+2-n)-(n^2+1-2ni+2n-2i))+\sum_{i=\frac{n}{4}+\frac{3}{2}}^{\frac{n}{2}-1}\alpha_{(i,\frac{n}{2})}((-1-n+2ni)-(-1+2i+2ni-2n))+\sum_{i=\frac{n}{4}+\frac{3}{2}}^{\frac{n}{2}-1}\gamma_{(i,\frac{n}{2})}((2ni+2-n)-(n^2-2ni+2i))+\sum_{i=1}^{\frac{n}{4}+\frac{1}{2}}\beta_{(i,\frac{n}{2})}((n^2-2ni+n)-(2ni-2i+2))+\sum_{i=1}^{\frac{n}{4}+\frac{1}{2}}\delta_{(i,\frac{n}{2})}((n^2+1-2ni+n)-(n^2+1-2ni+2n-2i))+\sum_{i=\frac{n}{4}+\frac{3}{2}}^{\frac{n}{2}-1}\beta_{(i,\frac{n}{2})}((n^2-2ni+n)-(-1+2i+2ni-2n))+\sum_{i=\frac{n}{4}+\frac{3}{2}}^{\frac{n}{2}-1}\delta_{(i,\frac{n}{2})}((n^2+1-2ni+n)-(n^2-2ni+2i)) \quad (15)$$

Fixed $n=8t+2$ and $\overrightarrow{\xi}=\overrightarrow{\xi}(n)\in\{1,2\}^{n^2-n}$, if substituting this into (15) results in a numerical equality, we will have found a magic square of order n if each row contains at most one non-zero coordinate of $\overrightarrow{\xi}$. If it results in a numerical difference, $\overrightarrow{\xi}$ will not form a magic square. If there is equality with the appearance of more than one non-zero element in any of the rows, $\overrightarrow{\xi}$ will construct a non-normal magic square. If there is a sequence of $\overrightarrow{\xi}(n), n=8t+2$ consistently forming magic squares, we will have discovered a method for constructing magic squares for all orders of the aforementioned type. We will apply the same definition to subsequences. Using the fundamental principle of counting, it is noted that there are at least $2^n\binom{n}{2}$ ways to use only two simple columns of (15) to ensure that there is no repetition of non-zero coordinates of $\overrightarrow{\xi}$.

Example 2. In (15), let's take $\alpha_{(i,j)}=\beta_{(i,j)}=\gamma_{(i,j)}=\delta_{(i,j)}=0=\alpha(i,\frac{n}{2})=\gamma(i,\frac{n}{2})$ to $1\le i\le\frac{n}{2};\ \ 1\le j\le\frac{n}{4}-\frac{1}{2};\ \ \frac{n}{4}+\frac{3}{2}\le j\le\frac{n}{2}-1$, and let's leave $\alpha_{(i,\frac{n}{4}+\frac{1}{2})}$; $\beta_{(i,\frac{n}{4}+\frac{1}{2})}$; $\gamma_{(i,\frac{n}{4}+\frac{1}{2})}$; $\delta_{(i,\frac{n}{4}+\frac{1}{2})}$; $\alpha_{(i,\frac{n}{2})}$, only, to be sought in order to satisfy (15). Here, we have the possibility of generating many magic squares swapping the elements of only one column. For example, if we restrict ourselves to only being able to swap the odd-numbered elements of the column of order $j=\frac{n}{4}+\frac{3}{2}$ with the elements of the main diagonal, we will have the following quadratic equation:

$$\sum_{i=1}^{\frac{n}{4}+\frac{1}{2}}\alpha_{(i,\frac{n}{4}+\frac{3}{2})}(2i-\tfrac{3n}{2})+\sum_{i=\frac{n}{4}+\frac{3}{2}}^{\frac{n}{2}-1}\alpha_{(i,\frac{n}{4}+\frac{3}{2})}(3-2i+\tfrac{n}{2})+\alpha_{(i,\frac{n}{4}+\frac{3}{2})}(3-\tfrac{n}{2})=-(\tfrac{n^2}{4}+n) \quad (16)$$

For n=10, (16) will be reduced to

$$-13\alpha_{(1,4)}-11\alpha_{(2,4)}-9\alpha_{(3,4)}-2\alpha_{(5,4)}-8\delta_{(1,5)}-6\delta_{(2,5)}-4\delta_{(3,5)}+3\delta_{(4,5)}=-35 \quad (17)$$

If in (17) we take $\alpha_{(1,4)}=\alpha_{(2,4)}=\alpha_{(3,4)}=\alpha_{(5,4)}=1$ and $\delta_{(1,5)}=\delta_{(2,5)}=\delta_{(3,5)}=\delta_{(4,5)}=0$, (17) will transform into a numerical equality, indicating that if we swap the elements of the semimagic square B with coordinates $(1,7),(3,7),(5,7)$ and $(9,7)$ by the elements of the main diagonal of the same line, B will transform into a magic square C. Let's look for natural numbers $x,z\in\mathrm{N}$ such that

$$2z+\sum_{i=1}^{x}\left(2i-\frac{3n}{2}\right)=-(\frac{n^2}{4}+n) \quad (18)$$

The equation (18) represents a hyperbolic paraboloid and, the magic squares that result from it, we will call them "hyperboloparaboloidal" magic squares. If in (18) we do $n=8y+2, y\in\mathrm{N}^*$, we will have the following solution in $\mathbb{Z}$:

$$x=2n_1+1; y=n_2; z=2n_1^2-12n_1n_2+8n_2^2+2n_2+1; n_1,n_2\in\mathbb{Z} \quad (19)$$

Setting $n_1=n_2=1$, we will $x=3$ and $z=1$.

$$A = \begin{pmatrix} 1 & 99 & 3 & 97 & 5 & 95 & 7 & 93 & 9 & 91 \\ 20 & 82 & 18 & 84 & 16 & 86 & 14 & 88 & 12 & 90 \\ 21 & 79 & 23 & 77 & 25 & 75 & 27 & 73 & 29 & 71 \\ 40 & 62 & 38 & 64 & 36 & 66 & 34 & 68 & 32 & 70 \\ 41 & 59 & 43 & 57 & 45 & 55 & 47 & 53 & 49 & 51 \\ 60 & 42 & 58 & 44 & 56 & 46 & 54 & 48 & 52 & 50 \\ 61 & 39 & 63 & 37 & 65 & 35 & 67 & 33 & 69 & 31 \\ 80 & 22 & 78 & 24 & 76 & 26 & 74 & 28 & 72 & 30 \\ 81 & 19 & 83 & 17 & 85 & 15 & 87 & 13 & 89 & 11 \\ 100 & 2 & 98 & 4 & 96 & 6 & 94 & 8 & 92 & 10 \end{pmatrix}$$

Initial matrix A, defined in equation (1)

$$B = \begin{pmatrix} 20 & 82 & 18 & 84 & 16 & 86 & 7[-13] & 93 & 9 & 90 \\ 1 & 99 & 3 & 97 & 5 & 95 & 14 & 88 & 12 & 91[-8] \\ 40 & 62 & 38 & 64 & 36 & 66 & 27[-11] & 73 & 29 & 70 \\ 21 & 79 & 23 & 77 & 25 & 75 & 34 & 68 & 32 & 71[-6] \\ 60 & 42 & 58 & 44 & 56 & 46 & 47[-9] & 53 & 49 & 50 \\ 41 & 59 & 43 & 57 & 45 & 55 & 54 & 48 & 52 & 51[-4] \\ 80 & 22 & 78 & 24 & 76 & 26 & 67[0] & 33 & 69 & 30 \\ 61 & 39 & 63 & 37 & 65 & 35 & 74 & 28 & 72 & 31[3] \\ 100 & 2 & 98 & 4 & 96 & 6 & 87[-2] & 13 & 89 & 10 \\ 81 & 19 & 83 & 17 & 85 & 15 & 94 & 8 & 92 & 11[0] \end{pmatrix}$$

Semimagic square B. Note in the brackets the difference in relation to the element of the main diagonal of the same row. To form a magic square here, we only need to swap the elements of the odd rows of B with the respective elements of the main diagonal of the same row.

$$C = \begin{pmatrix} 7 & 82 & 18 & 84 & 16 & 86 & 20[-13] & 93 & 9 & 90 \\ 1 & 99 & 3 & 97 & 5 & 95 & 12 & 88 & 14 & 91[-8] \\ 40 & 62 & 27 & 64 & 36 & 66 & 38[-11] & 73 & 29 & 70 \\ 21 & 79 & 23 & 77 & 25 & 75 & 34 & 68 & 32 & 71[-6] \\ 60 & 42 & 58 & 44 & 47 & 46 & 56[-9] & 53 & 49 & 50 \\ 41 & 59 & 43 & 57 & 54 & 55 & 45 & 48 & 52 & 51[-4] \\ 80 & 22 & 78 & 24 & 76 & 26 & 67[0] & 33 & 69 & 30 \\ 61 & 39 & 74 & 37 & 65 & 35 & 63 & 28 & 72 & 31[3] \\ 100 & 2 & 98 & 4 & 96 & 6 & 89[-2] & 13 & 87 & 10 \\ 94 & 19 & 83 & 17 & 85 & 15 & 81 & 8 & 92 & 11[0] \end{pmatrix}$$

"Hyperboloparaboloidal" magic square C resulting from $-13 - 11 - 9 - 2 = -35 = -(\frac{10^2}{4} + 10)$. To do this, after creating matrix B, we need to swap: 7 with 20; 27 with 38; 56 with 47 and 87 with 89. And also make the swaps in the complementary entries equidistant from the central horizontal axis of the matrix.

Comment 4. If we assume that the $n^2 - n$ variables $\alpha_{(i,j)}, \beta_{(i,j)}, \gamma_{(i,j)}, \delta_{(i,j)}$ traverse the closed interval $[0, 1] \subset \mathbb{R}$, then the equation (15) represents a Riemannian manifold with boundary contained in $\mathbb{R}^{n^2-n+1}$ and of dimension $n^2 - n$. When we fix values of n we obtain hyperplanes and when we fix the aforementioned variables we obtain parabolas. The magic squares derived from (15) correspond to some points on the boundary of this Riemannian manifold of parabolic and also linear nature.

Proposition 7. Starting from any magic square $C = (c_{u,v})_{u,v \in I_n}$ holder of vertical self-complementary property we can generate $((n-2)!)^{\frac{n}{2}}$ magic squares, including the one itself C.

Demonstration. The matrix C has a complementary property, therefore, we can freely permute two elements (entries) not belonging to the main diagonal of any of the $\frac{n}{2}$ first simple rows of C, along with their respective complementary swaps, generating another magic square. Now, since any permutation is a composition of these swaps of two elements, the result follows.

Comment 5. Using edge identification mapping, it's possible to place a matrix of n order on a two-dimensional torus. The torus has n main diagonals, n secondary diagonals, as well as n minor circles and

n major circles. The diagonals of the torus correspond to the broken diagonals of the "mapping matrix". These numbers, on the torus, correspond to directed paths. By traversing all these $4n$ periodic paths in parallel from all possible points (numbers, too), we can naturally construct 2^{n^2+2} matrices. Parallelism guarantees that the set of entries of each of these matrices contains I_{n^2} if the "mapping matrix" also contains I_{n^2}.

Comment 6. The increase in the number of magic or semimagic squares produced from A, defined in (1), can be done by noting that (These sums do not depend on the order of the double columns, j):

$$(A_{i,j})_{1,1} + (A_{i,j})_{1,2} = n^2;\ (A_{i,j})_{1,2} + (A_{i,j})_{2,2} = -4ni + 2n^2 + 2n + 1;\ (A_{i,j})_{2,2} + (A_{i,j})_{2,1} = n^2 + 2;$$
$$(A_{i,j})_{2,1} + (A_{i,j})_{1,1} = 4ni - 2n + 1 \quad (20)$$

Comment 7. The results presented in this subsection apply entirely to matrices of order $n = 4k$, as well. For matrices of order $n = m^2(4k + 2), k, m \in \mathbb{N}^*$, the possibility of partitioning the set I_{n^2} into several arithmetic progressions generates an abundance of magic squares.

Comment 8. Balancing the sums in the rows of an n-order matrix (containing all the numbers $1, 2, 3, ..., n^2$) by transposing numbers from the same column is a good procedure for constructing magic squares. The same idea applies to the columns. This leads to difficult combinatorial problems, the subset sum problem.

Conjecture 1. Every magic square of even order can be constructed from a matrix bearing self- complementary properties by the procedures presented above or through analogous procedures performed on matrices that have transversal self-complementary properties in the two-dimensional torus.

6 Conclusion

A new method for constructing magic squares of order $n = 8t - 2$ have been established. Some analogous results derived from this method also construct magic squares of even orders different from $n = 8t - 2$. The method indicated a very large number of magic squares constructed from the many permutations of the numbers within the same line that the vertical complementarity property allows without destroying the magic sum in the rows and columns, once a magic square is made ($(n-2)!^{\frac{n}{2}}$ different magic squares). This fact encouraged us to attempt to establish a project for the classification of all magic squares of even orders. A clear guideline for future research is to use the experience gained from this article to find methods for constructing magic squares for the remaining even orders of the type $n = 8t + 2$ and also for doubly even orders. There are other possibilities, such as researching: whether the geometry of quadrics can provide information about magic squares and vice versa; whether coordinate transformations T(i,j) can generate new magic squares; the mechanical properties that these magic squares model; whether simultaneous swaps of two parallel rows and two parallel columns equidistant from the center of the constructed magic squares generate more than $(n-2)!^{\frac{n}{2}}$ magic squares in total.

References

[1] Danielsson, H. (2022). Magic Squares. 931 pages. Available at https://magic-squares.info/info/book.html. Access on March 28, 2023.

[2] Danielsson, Holger., 2024, Magische Quadrate und ihre Konstruktion : Ein ausführlicher Überblick zu bekannten Konstruktionsverfahren, 1st ed. Springer Spektrum, 931 pages. https://www.magische-quadrate.de/magische-quadrate/

[3] de Oliveira Miranda, L.; Barbosa Bacelar Miranda, L., 2020a, Lohans' Magic Squares and the Gaussian Elimination Method, JNMS, 3(1), 31-36. DOI: https://doi.org/10.3126/jnms.v3i1.33001.

[4] de Oliveira Miranda, L. e Barbosa Bacelar Miranda, L., 2020b, Generalization of Dürer's Magic Square and New Methods for Doubly Even Magic Squares. JNMS, 3(2),13-15. DOI: https://doi.org/10.3126/jnms.v3i2.33955

[5] Miranda, L. de O., and Miranda, L. B. B., 2021, Establishing Infinite Methods of Construction of Magic Squares. Journal of Nepal Mathematical Society, 4(1), 19–22. https://doi.org/10.3126/jnms.v4i1.37108

[6] Walter Trump, 2005, Estimates of the number of magic squares, cubes, ... (hypercubes) - Number of normal magic squares. Estimates using magic series. Available at http://www.trump.de/magic-squares/estimates/regular.htm. Accessed on February 26, 2026.

[7] Manjunath, H., 2020, Magic Squares from Simple Squares and a Method like Conway's LUX. ResearchGate Preprint. DOI: 10.13140/RG.2.2.28322.86726

[8] Chong, J. Y., 2025, On the Constructions of CK(m)-Magic Squares of Singly Even Orders. Tunku Abdul Rahman University of Management and Technology. https://eprints.tarc.edu.my/id/eprint/33822/

[9] Rani, N., Mishra, V., and Sharma, S. R., 2023, Image encryption model based on novel magic square with differential encoding and chaotic map. Nonlinear Dynamics, 111(3), 2869-2893.

[10] Rani, N., 2025, Image ciphering algorithm based on novel ICMM and modified differential encoding. Physica Scripta, 100(4), 045211.