

On Value Distribution of Differential Polynomials of Meromorphic Functions

Dibyendu Banerjee¹, Ishita Ghosh²

¹Department of Mathematics, Visva-Bharati, Santiniketan-731235, India

²Department of Mathematics, Dr. Bhupendra Nath Dutta Smriti Mahavidyalaya, Hatgobindapur-713407, India

Correspondence to: Dibyendu Banerjee, Email: dibyendu192@rediffmail.com

Abstract: In a recent paper, Fan et. al. [3] showed that for a non-homogeneous differential polynomial formed with a non-constant meromorphic function f , $\Theta(a, \phi(f))$ is bounded above by 1. In this paper, we extend two results of [3] for differential polynomials formed with two meromorphic functions.

Keywords: Meromorphic function, Differential polynomial, Value distribution.

1 Introduction

Throughout the paper a meromorphic function always means meromorphic in the whole complex plane and we use the standard notations in value distribution theory {[4], [5], [6], [7]} $T(r, f)$, $N(r, f)$, $\overline{N}(r, f)$, $m(r, f)$ etc.

We denote by $S(r, f)$ any function satisfying $S(r, f) = o(T(r, f))$ as $r \rightarrow \infty$, which is outside of an exceptional set E with finite linear measure. A meromorphic function $a = a(z) (\neq 0, \infty)$ is said to be a small function of f if it satisfies $T(r, a) = S(r, f)$.

For a non-constant meromorphic function f , we define

$$\Theta(\infty, f) = 1 - \limsup_{r \rightarrow \infty} \frac{\overline{N}(r, f)}{T(r, f)}.$$

In [3], Fan et. al. constructed a differential monomial of a non-constant meromorphic function f as

$$M_j(f) = a_j(z)(f(z))^{n_{0j}}(f'(z))^{n_{1j}} \dots (f^{(k)}(z))^{n_{kj}},$$

where j be a positive integer, $n_{0j}, n_{1j}, \dots, n_{kj}$ be non-negative integers and $a_j(z)$ be a small function of f . Next they defined a differential polynomial in f as

$$\phi(f) = \sum_{j=0}^n M_j(f). \quad (1)$$

They also defined

$$\begin{aligned} d(M_j) &= \sum_{i=0}^k n_{ij} \text{ as the degree of } M_j(f), \\ d(\phi) &= \max_{0 \leq j \leq n} \{d(M_j)\} \text{ as the degree of } \phi(f) \text{ and} \\ \underline{d}(\phi) &= \min_{0 \leq j \leq n} \{d(M_j)\} \text{ as the lower degree of } \phi(f). \end{aligned}$$

With these notations they have proved the following theorems.

Theorem 1.1. Suppose that f is a transcendental meromorphic function with

$$N(r, f) + N\left(r, \frac{1}{f}\right) = S(r, f).$$

Let $\phi(f)$ be of the form (1) of degree $d(\phi) \geq 2$ and the lower degree $\underline{d}(\phi) \geq 1$. Assume that $\phi(f)$ does not reduce to a constant.

(i) If $\phi(f)$ is a homogeneous differential polynomial, then we have

$$\Theta(a, \phi(f)) = 0,$$

where $a(\not\equiv 0)$ is a small function of f .

(ii) If $\phi(f)$ is a non-homogeneous differential polynomial, then we have

$$\Theta(a, \phi(f)) \leq 1 - \frac{\underline{d}(\phi)}{d(\phi)},$$

where $a(\not\equiv 0)$ is a small function of f .

Theorem 1.2. Suppose that P is a polynomial with $\deg P \geq 3$ and f is a transcendental meromorphic function. Let α be a small function of f . If α is a constant, we also require that there exists a constant $A \neq \alpha$ such that $P(z) - A$ has a zero of multiplicity at least 3. Then, for any $0 < \epsilon < 1$, we have

$$T(r, f) \leq k\bar{N}\left(r, \frac{1}{P(f) - \alpha}\right) + S(r, f),$$

where if $P'(z)$ has only one zero, then $k = \frac{1}{\#\deg P - 2}$, if $P'(z)$ has two distinct zeroes a and b with $P(a) \neq P(b)$ and a is non-constant, then $k = \frac{1}{1-\epsilon}$; otherwise $k = 1$.

In this paper, it therefore seems reasonable to define a differential polynomial formed with two meromorphic functions and prove the analogues theorems of the above with this differential polynomial.

Let f and g be two non-constant meromorphic functions and j be a positive integer. Let $n_{0j}, n_{1j}, \dots, n_{kj}, m_{0j}, m_{1j}, \dots, m_{lj}$ be non-negative integers and let $a_j(z)$ be a small function of at least one of f or g . We call that

$$M_j(f, g) = a_j(z)(f(z))^{n_{0j}}(f'(z))^{n_{1j}} \dots (f^{(k)}(z))^{n_{kj}}(g(z))^{m_{0j}}(g'(z))^{m_{1j}} \dots (g^{(l)}(z))^{m_{lj}}$$

is a differential monomial of f and g .

We define $d_j(f) = \sum_{i=0}^k n_{ij}$, $d_j(g) = \sum_{i=0}^l m_{ij}$ and $d(M_j) = d_j(f) + d_j(g)$ (degree of $M_j(f, g)$). Next, we define a differential polynomial in f and g as a finite sum of such monomials, i.e.,

$$\phi(f, g) = \sum_{j=0}^p M_j(f, g). \tag{2}$$

We can rewrite this as

$$\begin{aligned} \phi(f, g) &= \sum_{j=0}^p a_j(z)(f(z))^{n_{0j}}(f'(z))^{n_{1j}} \dots (f^{(k)}(z))^{n_{kj}}(g(z))^{m_{0j}}(g'(z))^{m_{1j}} \dots (g^{(l)}(z))^{m_{lj}} \\ &= \sum_{j=0}^p a_j(z)(f(z))^{n_{0j}+n_{1j}+\dots+n_{kj}}(g(z))^{m_{0j}+m_{1j}+\dots+m_{lj}} \\ &\quad \left(\frac{f'(z)}{f(z)}\right)^{n_{1j}} \dots \left(\frac{f^{(k)}(z)}{f(z)}\right)^{n_{kj}} \left(\frac{g'(z)}{g(z)}\right)^{m_{1j}} \dots \left(\frac{g^{(l)}(z)}{g(z)}\right)^{m_{lj}} \\ &= \sum_{j=0}^p b_j(z)(f(z))^{n_{0j}+n_{1j}+\dots+n_{kj}}(g(z))^{m_{0j}+m_{1j}+\dots+m_{lj}} \\ &= \sum_{j=0}^p b_j(z)(f(z))^{d_j(f)}(g(z))^{d_j(g)}, \end{aligned} \tag{3}$$

$$\text{where } b_j(z) = a_j(z) \left(\frac{f'(z)}{f(z)} \right)^{n_{1j}} \cdots \left(\frac{f^{(k)}(z)}{f(z)} \right)^{n_{kj}} \left(\frac{g'(z)}{g(z)} \right)^{m_{1j}} \cdots \left(\frac{g^{(l)}(z)}{g(z)} \right)^{m_{lj}}. \quad (4)$$

We also define

$$\left. \begin{aligned} d(\phi(f, g)) &= \max_{0 \leq j \leq p} d(M_j), \text{ the degree of } \phi(f, g); \\ \underline{d}(\phi(f, g)) &= \min_{0 \leq j \leq p} d(M_j), \text{ the lower degree of } \phi(f, g); \\ d(f) &= \max_{0 \leq j \leq p} d_j(f), \text{ the degree of } f \text{ in } \phi(f, g); \\ \underline{d}(f) &= \min_{0 \leq j \leq p} d_j(f), \text{ the lower degree of } f \text{ in } \phi(f, g); \\ d(g) &= \max_{0 \leq j \leq p} d_j(g), \text{ the degree of } g \text{ in } \phi(f, g); \\ \underline{d}(g) &= \min_{0 \leq j \leq p} d_j(g), \text{ the lower degree of } g \text{ in } \phi(f, g). \end{aligned} \right\} \quad (5)$$

If $d(\phi(f, g)) = \underline{d}(\phi(f, g))$, then $\phi(f, g)$ is called homogeneous and otherwise it is called non-homogeneous. In particular, if there exists at least one $j (= j_1, \text{ say})$ such that $d(f) = d_{j_1}(f)$ and $d(g) = d_{j_1}(g)$, then we can write

$$d(\phi(f, g)) = d(f) + d(g). \quad (6)$$

Similarly if there exists at least one $j (= j_2, \text{ say})$ such that $\underline{d}(f) = \underline{d}_{j_2}(f)$ and $\underline{d}(g) = \underline{d}_{j_2}(g)$, then we can write

$$\underline{d}(\phi(f, g)) = \underline{d}(f) + \underline{d}(g). \quad (7)$$

To clarify this particular scenario we can consider the following examples.

Example 1. Let, $\phi(f, g) = b_0 f^2 g^3 + b_1 f g^4 + b_2 g^5 + b_3 f^4 g$. We may write this as

$$\phi(f, g) = \sum_{j=0}^3 b_j f^{d_j(f)} g^{d_j(g)}, \text{ where } d_0(f) = 2, d_0(g) = 3, d_1(f) = 1, d_1(g) = 4, d_2(f) = 0, d_2(g) = 5, d_3(f) = 4, d_3(g) = 1.$$

Here

$$\begin{aligned} d(\phi(f, g)) &= 5 \\ \underline{d}(\phi(f, g)) &= 5 \\ d(f) &= \max\{2, 1, 0, 4\} = 4 \\ \underline{d}(f) &= \min\{2, 1, 0, 4\} = 0 \\ d(g) &= \max\{3, 4, 5, 1\} = 5 \\ \underline{d}(g) &= \min\{3, 4, 5, 1\} = 1 \end{aligned}$$

Here the polynomial is homogeneous, since $d(\phi(f, g)) = \underline{d}(\phi(f, g))$. But we see that $d(\phi(f, g)) \leq d(f) + d(g)$ and $\underline{d}(\phi(f, g)) \geq \underline{d}(f) + \underline{d}(g)$ i.e., (6) and (7) are not satisfied in this case.

Example 2. Let, $\phi(f, g) = b_0 f^4 g^4 + b_1 f^5 g^4 + b_2 f^4 g^3$. We may write this as

$$\phi(f, g) = \sum_{j=0}^2 b_j f^{d_j(f)} g^{d_j(g)}, \text{ where } d_0(f) = 4, d_0(g) = 4, d_1(f) = 5, d_1(g) = 4, d_2(f) = 4, d_2(g) = 3.$$

Here

$$\begin{aligned} d(\phi(f, g)) &= \max\{8, 9, 7\} = 9; \\ \underline{d}(\phi(f, g)) &= \min\{8, 9, 7\} = 7; \\ d(f) &= \max\{4, 5, 4\} = 5; \\ \underline{d}(f) &= \min\{4, 5, 4\} = 4; \\ d(g) &= \max\{4, 4, 3\} = 4; \\ \underline{d}(g) &= \min\{4, 4, 3\} = 3. \end{aligned}$$

Also, we can see that $j_1 = 1$, which means both $d(f)$ and $d(g)$ are attained in the second term of the differential polynomial and $j_2 = 2$, which means both $\underline{d}(f)$ and $\underline{d}(g)$ are attained in the third term of

the differential polynomial. Hence for this differential polynomial, (6) and (7) are satisfied although the differential polynomial is not homogeneous.

Example 3. Let, $\phi(f, g) = b_0fg^2 + b_1f^2g^2 + b_2g^3 + b_3f^2 + b_4f^4g$. We may write this as

$\phi(f, g) = \sum_{j=0}^4 b_j f^{d(f_j)} g^{d(g_j)}$, where $d_0(f) = 1, d_1(f) = 2, d_2(f) = 0, d_3(f) = 2, d_4(f) = 4, d_0(g) = 2, d_1(g) = 2, d_2(g) = 3, d_3(g) = 0, d_4(g) = 1$. So,

$$\begin{aligned} \phi(f, g) &= b_0fg^2 + b_1f^2g^2 + b_2g^3 + b_3f^2 + b_4f^4g \\ &= b_2g^3 + f(b_0g^2) + f^2(b_1g^2 + b_3) + f^4(b_4g) \\ &= f^0(b_{00}g^2 + b_{01}g^2 + b_{02}g^3 + b_{03}g^0 + b_{04}g^1) + f^1(b_{10}g^2 + b_{11}g^2 + b_{12}g^3 + b_{13}g^0 + b_{14}g^1) + \\ &\quad f^2(b_{20}g^2 + b_{21}g^2 + b_{22}g^3 + b_{23}g^0 + b_{24}g^1) + f^4(b_{30}g^2 + b_{31}g^2 + b_{32}g^3 + b_{33}g^0 + b_{34}g^1), \\ &\quad \text{where } (b_{ij})_{4 \times 5} = \begin{pmatrix} 0 & 0 & b_2 & 0 & 0 \\ b_0 & 0 & 0 & 0 & 0 \\ 0 & b_1 & 0 & b_3 & 0 \\ 0 & 0 & 0 & 0 & b_4 \end{pmatrix} \\ &= f^{u_0} \sum_{j=0}^0 c_{0j} g^{v_{0j}} + f^{u_1} \sum_{j=0}^0 c_{1j} g^{v_{1j}} + f^{u_2} \sum_{j=0}^1 c_{2j} g^{v_{2j}} + f^{u_3} \sum_{j=0}^0 c_{3j} g^{v_{3j}}. \end{aligned} \quad (8)$$

Here $u_0 = 0, u_1 = 1, u_2 = 2, u_3 = 4$. Also we remove all b_{0j} 's, b_{1j} 's, b_{2j} 's, b_{3j} 's etc., which are zero and rename non-zero terms b_{0j} 's, b_{1j} 's, b_{2j} 's, b_{3j} 's etc., as c_{0j} 's, c_{1j} 's, c_{2j} 's, c_{3j} 's etc., respectively, i.e., $b_{02} = b_2 = c_{00}, b_{10} = b_0 = c_{10}, b_{21} = b_1 = c_{20}, b_{23} = b_3 = c_{21}, b_{34} = b_4 = c_{30}$ and $v_{00} = d_2(g) = 3, v_{10} = d_0(g) = 2, v_{20} = d_1(g) = 0, v_{21} = d_3(g) = 2, v_{30} = d_4(g) = 1$.

Here it is quite clear that $\min\{v_{00}, v_{10}, v_{20}, v_{30}\} = 0$, which is same as the lower degree of g defined in (5) and $\max\{v_{00}, v_{10}, v_{21}, v_{30}\} = 3$, which is same as the degree of g defined in (5).

So, from (8) we understand that in general (3) can be written as

$$\phi(f, g) = \sum_{i=0}^q f^{u_i} \sum_{j=0}^{p_i} c_{ij} g^{v_{ij}}, \text{ where} \quad (9)$$

$q \leq p, u_0, u_1, \dots, u_q$ are non-negative integers with $\underline{d}(f) = u_0 < u_1 < \dots < u_q = d(f)$, and $p_i \leq p$ ($i = 0, 1, \dots, q$), $c_{ij} \neq 0, v_{ij}$'s are positive integers with $v_{i0} < v_{i1} < \dots < v_{ip_i}$.

Again we also note that $\underline{d}(g) = \min\{v_{00}, v_{10}, \dots, v_{q0}\} = v_t$ (say) and $d(g) = \max\{v_{0p_0}, v_{1p_1}, \dots, v_{qp_q}\} = v_T$ (say).

In a recent paper [2], Banerjee and Pal introducing the definition of a differential polynomial which involve two distinct transcendental meromorphic functions, proved some results on value distribution of differential polynomials.

2 Known Results

In this section we state some results in the form of lemmas, which will be needed in the subsequent section.

Lemma 2.0.1. $\{[4], [5], [7]\}$ Let f be a non-constant meromorphic function, and let k be a positive integer. Then

$$m\left(r, \frac{f^{(k)}}{f}\right) = S(r, f).$$

Lemma 2.0.2. $\{[4], [5], [7]\}$ Let f be a non-constant meromorphic function, and let $a_i (i = 1, 2)$ be two distinct small functions of f . Then

$$T(r, f) \leq \bar{N}(r, f) + \bar{N}\left(r, \frac{1}{f - a_1}\right) + \bar{N}\left(r, \frac{1}{f - a_2}\right) + S(r, f).$$

Lemma 2.0.3. [1] Let f and g be two transcendental meromorphic functions and a_i , $1 \leq i \leq (p+1)(q+1)$ are small functions of at least f or g . Set

$$P_{p,q}(f, g) = \sum_{i=1}^{p+1} \sum_{j=1}^{q+1} a_{(i-1)(q+1)+j} f^{p-i+1} g^{q-j+1}.$$

If $N(r, f) = S(r, f)$, $N(r, g) = S(r, g)$ with $a_1 \neq 0$, then

$$T(r, P_{p,q}(f, g)) \geq pT(r, f) + qT(r, g) + S(r, f) + S(r, g).$$

3 Theorems

In this section we present our main results of the paper.

Theorem 3.1. Suppose that f and g are two transcendental meromorphic functions. Let $\phi(f, g)$ be a differential polynomial in f and g of the form (2) of degree $d(\phi) \geq 2$ and the lower degree $\underline{d}(\phi) \geq 1$. Assume that $\phi(f, g)$ does not reduce to a constant. Then,

(i) if $\phi(f, g)$ is homogeneous with

$$N(r, f) + N(r, \frac{1}{f}) = S(r, f), \quad N(r, g) + N(r, \frac{1}{g}) = S(r, g) \tag{10}$$

and satisfies at least one of (6) or (7), then we have

$$\Theta(a, \phi(f, g)) = 0,$$

where $a(\neq 0)$ is a small function of at least f or g .

(ii) if $\phi(f, g)$ is other than (i) and satisfies $N(r, f) = S(r, f)$, $N(r, g) = S(r, g)$ and $\frac{d(f)}{d(f)} = \frac{d(g)}{d(g)} = A$ (say), then we have

$$\Theta(a, \phi(f, g)) \leq 1 - A,$$

where $a(\neq 0)$ is a small function of at least f or g .

Proof. It is clear that $\phi(f, g) \neq 0$.

(i) $\phi(f, g)$ is homogeneous and satisfies at least one of (6) or (7). Then, by (3) we have

$$\phi(f, g) = B(z)(f(z))^{d(f)}(g(z))^{d(g)}, \tag{11}$$

where $B(z) = \sum_{j=0}^p b_j(z)$.

Since $a_j(z)$ is a small function of at least one of f or g , by Lemma 2.0.1 and (10), we obtain from (4)

$$\begin{aligned} m(r, b_j) &\leq m(r, a_j) + m\left(r, \left(\frac{f'}{f}\right)^{n_{1j}} \cdots \left(\frac{f^{(k)}}{f}\right)^{n_{kj}}\right) + m\left(r, \left(\frac{g'}{g}\right)^{m_{1j}} \cdots \left(\frac{g^{(k)}}{g}\right)^{m_{kj}}\right) \\ &= S(r, f) + S(r, g), \end{aligned}$$

and

$$\begin{aligned} N(r, b_j) &\leq N(r, a_j) + N\left(r, \left(\frac{f'}{f}\right)^{n_{1j}} \cdots \left(\frac{f^{(k)}}{f}\right)^{n_{kj}}\right) + N\left(r, \left(\frac{g'}{g}\right)^{m_{1j}} \cdots \left(\frac{g^{(k)}}{g}\right)^{m_{kj}}\right) \\ &\leq N(r, a_j) + O\left(N(r, f) + N\left(r, \frac{1}{f}\right)\right) + O\left(N(r, g) + N\left(r, \frac{1}{g}\right)\right) \\ &= S(r, f) + S(r, g). \end{aligned}$$

Hence we obtain

$$\begin{aligned} T(r, b_j) &= S(r, f) + S(r, g) \\ \text{i.e., } T(r, B) &= S(r, f) + S(r, g). \end{aligned} \quad (12)$$

By (10), (11), (12) and Lemma 2.0.2, we have

$$\begin{aligned} T(r, \phi(f, g)) &\leq \bar{N}(r, \phi(f, g)) + \bar{N}\left(r, \frac{1}{\phi(f, g)}\right) + \bar{N}\left(r, \frac{1}{\phi(f, g) - a}\right) + S(r, \phi(f, g)) \\ &\leq \bar{N}(r, Bf^{d(f)}g^{d(g)}) + \bar{N}\left(r, \frac{1}{Bf^{d(f)}g^{d(g)}}\right) + \bar{N}\left(r, \frac{1}{\phi(f, g) - a}\right) + S(r, \phi(f, g)) \\ &= S(r, f) + S(r, g) + \bar{N}\left(r, \frac{1}{\phi(f, g) - a}\right) + S(r, \phi(f, g)) \\ &= \bar{N}\left(r, \frac{1}{\phi(f, g) - a}\right) + S(r, \phi(f, g)). \end{aligned} \quad (13)$$

Hence from (13), we obtain $\Theta(a, \phi(f, g)) = 0$.

(ii) $\phi(f, g)$ is other than (i) and satisfies $N(r, f) = S(r, f)$, $N(r, g) = S(r, g)$ and $\frac{d(f)}{d(f)} = \frac{d(g)}{d(g)} = A$. Then from (9) by using similar arguments as (i) we can easily obtain similar results as (12), i.e.,

$$T(r, c_{ij}) = S(r, f) + S(r, g). \quad (14)$$

By Lemma 2.0.3, we obtain

$$T(r, \phi(f, g)) \geq d(f)T(r, f) + d(g)T(r, g) + S(r, f) + S(r, g). \quad (15)$$

Also using (9) we obtain

$$\begin{aligned} \bar{N}\left(r, \frac{1}{\phi(f, g)}\right) &\leq \bar{N}\left(r, \frac{1}{f^{u_0}g^{v_t} \left[\sum_{i=0}^q f^{u_i - u_0} \sum_{j=0}^{p_i} c_{ij}g^{v_{ij} - v_t} \right]}\right) \\ &\leq \bar{N}\left(r, \frac{1}{f^{u_0}g^{v_t}}\right) + T\left(r, \sum_{i=0}^q f^{u_i - u_0} \sum_{j=0}^{p_i} c_{ij}g^{v_{ij} - v_t}\right) \\ &\leq S(r, f) + S(r, g) + (u_q - u_0)T(r, f) + (v_T - v_t)T(r, g) \\ &\leq (d(f) - \underline{d}(f))T(r, f) + (d(g) - \underline{d}(g))T(r, g) + S(r, \phi(f, g)). \end{aligned} \quad (16)$$

Again $\frac{d(f)}{d(f)} = \frac{d(g)}{d(g)} = A$ gives

$$\begin{aligned} 1 - \frac{\underline{d}(f)}{d(f)} &= 1 - \frac{\underline{d}(g)}{d(g)} = 1 - A \\ \text{i.e., } \frac{d(f) - \underline{d}(f)}{d(f)} &= \frac{d(g) - \underline{d}(g)}{d(g)} = 1 - A \\ \text{i.e., } d(f) - \underline{d}(f) &= (1 - A)d(f) \text{ and } d(g) - \underline{d}(g) = (1 - A)d(g). \end{aligned} \quad (17)$$

(15), (16) and (17) gives

$$\begin{aligned} \bar{N}\left(r, \frac{1}{\phi(f, g)}\right) &\leq (1 - A)d(f)T(r, f) + (1 - A)d(g)T(r, g) + S(r, \phi(f, g)) \\ &\leq (1 - A)T(r, \phi(f, g)) + S(r, \phi(f, g)). \end{aligned} \quad (18)$$

It follows from the given conditions and from (9), (10), (14), (18) and Lemma 2.0.2 that

$$\begin{aligned} T(r, \phi(f, g)) &\leq \overline{N}(r, \phi(f, g)) + \overline{N}\left(r, \frac{1}{\phi(f, g)}\right) + \overline{N}\left(r, \frac{1}{\phi(f, g) - a}\right) + S(r, \phi(f, g)) \\ &\leq (1 - A)T(r, \phi(f, g)) + \overline{N}\left(r, \frac{1}{\phi(f, g) - a}\right) + S(r, \phi(f, g)). \end{aligned} \tag{19}$$

Hence from (19), we have $\Theta(a, \phi(f, g)) \leq 1 - A$.

Hence the theorem. □

Remark 3.1. If $d_0(g) = d_1(g) = \dots = d_p(g) = 0$ in (3), then $d(g) = \underline{d}(g) = 0$. So $\phi(f, g)$ reduces to $\phi(f)$ like (1). In this case Theorem 3.1 reduces to Theorem 1.1 by Fan et. al. [3].

Theorem 3.2. Suppose P, Q are polynomials with degrees $m \geq 3, n \geq 3$ respectively and f, g are transcendental meromorphic functions. Set $R(f, g) = P(f)Q(g)$, which is a polynomial with degree $m + n$. Let α be a small function of f and g .

(i) If $P'(z)$ and $Q'(z)$ have only a zero at a and b respectively with $P(a) = Q(b) = 0$, then $(m - 2)T(r, f) + (n - 2)T(r, g) \leq \overline{N}\left(r, \frac{1}{R(f, g) - \alpha}\right) + S(r, R(f, g))$.

(ii) If $P'(z)$ and $Q'(z)$ have at least two distinct zeros at a_p, b_p and a_q, b_q respectively with $P(a_p) = P(b_p) = Q(a_q) = Q(b_q) = 0$, then

$$T(r, f) + T(r, g) \leq \overline{N}\left(r, \frac{1}{R(f, g) - \alpha}\right) + S(r, R(f, g)).$$

Proof. (i) Let $P'(z)$ and $Q'(z)$ have only a zero at a and b respectively with $P(a) = Q(b) = 0$. Then we have

$$P(z) = c_1(z - a)^m, \quad Q(z) = c_2(z - b)^n, \tag{20}$$

where c_1, c_2 are non-zero constants.

By Lemma 2.0.2 and (20), we obtain

$$\begin{aligned} T(r, R(f, g)) &\leq \overline{N}(r, R(f, g)) + \overline{N}\left(r, \frac{1}{R(f, g)}\right) + \overline{N}\left(r, \frac{1}{R(f, g) - \alpha}\right) + S(r, R(f, g)) \\ &\leq \overline{N}(r, f) + \overline{N}(r, g) + \overline{N}\left(r, \frac{1}{f - a}\right) + \overline{N}\left(r, \frac{1}{g - b}\right) + \overline{N}\left(r, \frac{1}{R(f, g) - \alpha}\right) + \\ &\hspace{15em} S(r, R(f, g)) \\ &\leq 2T(r, f) + 2T(r, g) + \overline{N}\left(r, \frac{1}{R(f, g) - \alpha}\right) + S(r, R(f, g)). \end{aligned} \tag{21}$$

Therefore from Lemma 2.0.3 and (21) that

$$(m - 2)T(r, f) + (n - 2)T(r, g) \leq \overline{N}\left(r, \frac{1}{R(f, g) - \alpha}\right) + S(r, R(f, g)).$$

(ii) Let, $P'(z)$ and $Q'(z)$ have at least two distinct zeros at a_p, b_p and a_q, b_q respectively with $P(a_p) = P(b_p) = Q(a_q) = Q(b_q) = 0$. Also let the order of zeros at a_p, b_p, a_q and b_q are m_a, m_b, n_a and n_b respectively. Then we have

$$P(z) = c_3(z - a_p)^{m_a+1}(z - b_p)^{m_b+1}\phi_1(z), \quad Q(z) = c_4(z - a_q)^{n_a+1}(z - b_q)^{n_b+1}\phi_2(z), \tag{22}$$

where c_3, c_4 are nonzero constants and $\phi_1(z), \phi_2(z)$ are polynomials with $\deg(\phi_1) = m - m_a - m_b - 2$, $\deg(\phi_2) = n - n_a - n_b - 2$ respectively.

By Lemma 2.0.2 and (22), we obtain

$$\begin{aligned}
 T(r, R(f, g)) &\leq \bar{N}(r, R(f, g)) + \bar{N}\left(r, \frac{1}{R(f, g)}\right) + \bar{N}\left(r, \frac{1}{R(f, g) - \alpha}\right) + S(r, R(f, g)) \\
 &\leq \bar{N}(r, f) + \bar{N}(r, g) + \bar{N}\left(r, \frac{1}{f - a_p}\right) + \bar{N}\left(r, \frac{1}{f - b_p}\right) + (m - m_a - m_b - 2)T(r, f) + \\
 &\quad \bar{N}\left(r, \frac{1}{g - a_q}\right) + \bar{N}\left(r, \frac{1}{g - b_q}\right) + (n - n_a - n_b - 2)T(r, g) + \\
 &\quad \bar{N}\left(r, \frac{1}{R(f, g) - \alpha}\right) + S(r, R(f, g)) \\
 &\leq (m - m_a - m_b + 1)T(r, f) + (n - n_a - n_b + 1)T(r, g) + \bar{N}\left(r, \frac{1}{R(f, g) - \alpha}\right) + \\
 &\quad S(r, R(f, g)) \\
 &\leq (m - 1)T(r, f) + (n - 1)T(r, g) + \bar{N}\left(r, \frac{1}{R(f, g) - \alpha}\right) + S(r, R(f, g)). \tag{23}
 \end{aligned}$$

Therefore from Lemma 2.0.3 and (23) that

$$T(r, f) + T(r, g) \leq \bar{N}\left(r, \frac{1}{R(f, g) - \alpha}\right) + S(r, R(f, g)).$$

□

Remark 3.2. *If $Q(g) = 1$ in (3), then $R(f, g) = P(f)$. In this case Theorem 3.2 reduces to Theorem 1.2 by Fan et. al. [3].*

4 Conclusions

Theorem 3.1 may be extended for differential polynomials formed with any finite number of meromorphic functions with certain restrictions.

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