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Some Identities Involving Central Binomial Coefficients and Nielsen's Beta Function

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Abstract: *This paper studies infinite series involving Nielsen's beta function and the central binomial coefficients* $\binom{2\ell}{\ell}$ *and* $\binom{3\ell}{\ell}$. *Using integral representations of Nielsen's beta function, we obtain closed-form evaluations of several infinite series. As applications, we derive new series representations for Apéry's constant, Catalan's constant, and the logarithmic constant* $\ln(2)$. *The resulting identities establish new connections between special functions, binomial coefficients, infinite series, and classical constants.*

Keywords: Infinite series, Binomial coefficients, Nielsen's beta function.

1 Introduction

Infinite sums comprising central binomial coefficients and harmonic numbers are highly significant across combinatorics and special functions. In particular, sums with binomial weights $\binom{2\ell}{\ell}$ and $\binom{3\ell}{\ell}$ often yield elegant closed forms obtained through integral and generating function methods; see the recent references [17], [3], and [1]. Such series are closely related to classical constants and special functions: see [2, 5, 9, 19, 6] and the references therein.

However, research on infinite series involving the Nielsen beta function remains limited. Recently, the sum of an infinite series involving the Nielsen beta function was investigated in [18]. In particular, considerable attention has been devoted to the monotonicity and bounds of Nielsen's beta function; (see [11, 12, 13]). In this paper, we derive the sums of infinite series involving Nielsen's beta function and binomial coefficients. For the reader's convenience, we first recall some basic properties of Nielsen's $\mathfrak{B}$ function originally introduced in [16], and defined in a variety of identical forms:

$$\mathfrak{B}(\tau) = \int_0^1 \frac{\vartheta^{\tau-1}}{1+\vartheta}\, d\vartheta = \int_0^\infty \frac{e^{-\tau\vartheta}}{1+e^{-\vartheta}}\, d\vartheta, \tag{1}$$

$$\mathfrak{B}(\tau) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell+\tau} = \frac{1}{2}\left\{\Psi\left(\frac{\tau+1}{2}\right) - \Psi\left(\frac{\tau}{2}\right)\right\}, \tau > 0, \tag{2}$$

where $\Psi(\tau) = \frac{d}{d\tau}\ln\Gamma(\tau)$ is the psi(digamma) function. One recalls that $\mathfrak{B}$ obeys the following recursive functional equation (see [16, 14]):

$$\mathfrak{B}(\tau+1) + \mathfrak{B}(\tau) = \frac{1}{\tau}, \tau > 0. \tag{3}$$

A few notable evaluations and particular values for $\mathfrak{B}$ are detailed below: (see [14]):

$$\mathfrak{B}(1) = \ln(2), \quad \mathfrak{B}\left(\tfrac{1}{2}\right) = \pi/2, \quad \mathfrak{B}\left(\tfrac{3}{2}\right) = 2 - \pi/2, \quad \text{and} \quad \mathfrak{B}(2) = 1 - \ln(2).$$

Recent developments on the Nielsen beta function can be found in [11, 12, 15, 4]. The special value of the Riemann zeta function at $s = 3$, namely

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3},$$

is referred to as Apéry's constant. The Catalan constant is defined by:

$$G=\sum_{\ell=0}^{\infty}\frac{(-1)^{\ell}}{(2\ell+1)^2}.$$

The motivation of this paper is to bridge the gap by deriving closed-form identities that explicitly bind the Nielsen beta function together with central binomial coefficients.

The paper proceeds as follows. Section 1 is devoted to introduction and preliminaries. Section 2 establishes the main results. In Section 3, we derive generating functions for the sequences $\binom{2\ell}{\ell}\mathfrak{B}(\ell+1)$ and $\ell\binom{2\ell}{\ell}\mathfrak{B}(\ell+1)$. Section 4, provides a discussion of the obtained results. Finally, Section 5 presents the conclusions and proposes an open problem.

2 Sums of infinite series involving Nielsen's $\mathfrak{B}$ function

We are in a position to derive some identities for infinite series involving Nielsen's $\mathfrak{B}$ function.
Theorem 2.1. The following equality holds:

$$\sum_{\ell=1}^{\infty}\frac{\mathfrak{B}(\ell)}{\ell}=\frac{1}{2}\ln^2(2)+\frac{\pi^2}{12}.$$

Proof. Using (1) and the identity $\frac{1}{\ell}=\int_0^1\tau^{\ell-1}\,d\tau$, the series becomes

$$\sum_{\ell=1}^{\infty}\frac{\mathfrak{B}(\ell)}{\ell}=\sum_{\ell=1}^{\infty}\int_0^1\int_0^1\frac{(\tau t)^{\ell-1}}{1+t}\,d\tau\,dt.$$

The non-negativity of the terms $\frac{(\tau t)^{\ell-1}}{1+t}$ on the domain $(0,1)^2$ permits the use of Tonelli's theorem to reverse the summation and integration signs, yielding:

$$\begin{aligned}\sum_{\ell=1}^{\infty}\frac{\mathfrak{B}(\ell)}{\ell}&=\int_0^1\int_0^1\frac{1}{1+t}\sum_{\ell=1}^{\infty}(\tau t)^{\ell-1}\,d\tau\,dt\\&=\int_0^1\int_0^1\frac{d\tau\,dt}{(1+t)(1-\tau t)}=-\int_0^1\frac{1}{t(1+t)}\left[\int_0^1\frac{-t}{1-\tau t}\,d\tau\right]dt=-\int_0^1\frac{\ln(1-t)}{t(1+t)}\,dt\\&=\int_0^1\frac{\ln(1-t)}{1+t}\,dt-\int_0^1\frac{\ln(1-t)}{t}\,dt.\end{aligned}$$

According to [10, Entry 4.291(2)], the second integral evaluates to

$$\int_0^1\frac{\ln(1-t)}{t}\,dt=-\frac{\pi^2}{6}. \tag{4}$$

Upon setting $u=(1-t)/(1+t)$, we find:

$$\int_0^1\frac{\ln(1-t)}{1+t}\,dt=\int_0^1\frac{\ln(2)}{1+u}\,du+\int_0^1\frac{\ln(u)}{1+u}\,du-\int_0^1\frac{\ln(1+u)}{1+u}du.$$

According to [10, Entry 4.231(1)], we have

$$\int_0^1\frac{\ln(u)}{1+u}\,du=-\frac{\pi^2}{12}.$$

Therefore,

$$\int_0^1\frac{\ln(1-t)}{1+t}\,dt=\ln^2(2)-\frac{\pi^2}{12}-\frac{1}{2}\ln^2(2)=\frac{1}{2}\ln^2(2)-\frac{\pi^2}{12}. \tag{5}$$

Thus, by combining equations (4) and (5), we obtain

$$\sum_{\ell=1}^{\infty} \frac{\mathfrak{B}(\ell)}{\ell} = \left(\frac{1}{2}\ln^2(2) - \frac{\pi^2}{12}\right) - \left(-\frac{\pi^2}{6}\right) = \frac{1}{2}\ln^2(2) + \frac{\pi^2}{12}.$$

□

We now examine alternating versions of these series and derive the corresponding identities.

Theorem 2.2. The following identity holds:

$$\sum_{\ell=1}^{\infty} (-1)^{\ell-1} \frac{\mathfrak{B}(\ell)}{\ell} = \frac{\pi^2}{12} - \frac{1}{2}\ln^2(2).$$

Proof. By (1) and the identity $\frac{1}{\ell} = \int_0^1 \tau^{\ell-1} d\tau$, the series can be expressed as

$$\sum_{\ell=1}^{\infty} (-1)^{\ell-1} \frac{\mathfrak{B}(\ell)}{\ell} = \sum_{\ell=1}^{\infty} (-1)^{\ell-1} \int_0^1 \int_0^1 \frac{(\tau t)^{\ell-1}}{1+t}\, d\tau\, dt.$$

Using the same argument as in Theorem 2.1, we transpose the infinite summation and integration to obtain,

$$\begin{aligned}\sum_{\ell=1}^{\infty} (-1)^{\ell-1} \frac{\mathfrak{B}(\ell)}{\ell} &= \int_0^1 \int_0^1 \frac{1}{1+t} \sum_{k=1}^{\infty} (-\tau t)^{\ell-1}\, d\tau\, dt \\ &= \int_0^1 \int_0^1 \frac{1}{(1+t)(1+\tau t)}\, d\tau\, dt = \int_0^1 \frac{1}{1+t} \left[\frac{\ln(1+\tau t)}{t}\right]_{\tau=0}^{\tau=1} dt = \int_0^1 \frac{\ln(1+t)}{t(1+t)}\, dt.\end{aligned}$$

whence,

$$\sum_{\ell=0}^{\infty} (-1)^{\ell} \frac{\mathfrak{B}(\ell+1)}{\ell+1} = \int_0^1 \frac{\ln(1+t)}{t}\, dt - \int_0^1 \frac{\ln(1+t)}{1+t}\, dt.$$

According to [10, Entry 4.291(2)], the first integral evaluates to

$$\int_0^1 \frac{\ln(1+t)}{t}\, dt = \frac{\pi^2}{12}. \tag{6}$$

The second integral is elementary:

$$\int_0^1 \frac{\ln(1+t)}{1+t}\, dt = \left[\frac{1}{2}\ln^2(1+t)\right]_0^1 = \frac{1}{2}\ln^2(2).$$

Combining these results, we find

$$\sum_{\ell=1}^{\infty} (-1)^{\ell-1} \frac{\mathfrak{B}(\ell)}{\ell} = \frac{\pi^2}{12} - \frac{1}{2}\ln^2(2).$$

□

By replacing the denominators with odd integers, we obtain new series identities connected to classical constants such as Catalan's constant.

Theorem 2.3. Let G denote Catalan's constant. Then, the following identity holds:

$$\sum_{\ell=1}^{\infty} \frac{\mathfrak{B}(\ell)}{2\ell-1} = G.$$

Proof. Using (1) and the identity $\frac{1}{\ell+1} = \int_0^1 \tau^\ell d\tau$, the series can be expressed as

$$\sum_{\ell=0}^{\infty} \frac{\mathfrak{B}(\ell+1)}{2\ell+1} = \sum_{\ell=0}^{\infty} \int_0^1 (\tau)^{2\ell}\, d\tau \int_0^1 \frac{t^\ell}{1+t} dt.$$

Using the same argument as in Theorem 2.1, we transpose the infinite summation and integration, obtaining

$$\begin{aligned}\sum_{\ell=0}^{\infty} \frac{\mathfrak{B}(\ell+1)}{2\ell+1} &= \int_0^1 \int_0^1 \frac{1}{1+t} \sum_{\ell=0}^{\infty} (\tau^2 t)^\ell \, d\tau\, dt \\ &= \int_0^1 \int_0^1 \frac{1}{(1+t)(1-\tau^2 t)} \, d\tau\, dt.\end{aligned}$$

Evaluating the inner integral with τ yields

$$\int_0^1 \frac{1}{1+t} \left[\frac{1}{2\sqrt{t}} \ln\left(\frac{1+\sqrt{t}\,\tau}{1-\sqrt{t}\,\tau} \right) \right]_{\tau=0}^{\tau=1} dt = \frac{1}{2} \int_0^1 \frac{1}{\sqrt{t}(1+t)} \ln\left(\frac{1+\sqrt{t}}{1-\sqrt{t}} \right) \, dt.$$

Applying the substitution $t = u^2$ simplifies the integral to

$$\frac{1}{2} \int_0^1 \frac{1}{\sqrt{t}(1+t)} \ln\left(\frac{1+\sqrt{t}}{1-\sqrt{t}} \right) \, dt = \int_0^1 \left(\frac{\ln(1+u)}{1+u^2} - \frac{\ln(1-u)}{1+u^2} \right) du.$$

According to [10, Entry 4.291(8 and 10)], the integrals evaluate to

$$\int_0^1 \frac{\ln(1+u)}{1+u^2}\, du = \frac{\pi}{8} \ln(2) \quad \text{and} \quad \int_0^1 \frac{\ln(1-u)}{1+u^2}\, du = \frac{\pi}{8} \ln(2) - G. \tag{7}$$

Combining these results, we find

$$\sum_{\ell=0}^{\infty} \frac{\mathfrak{B}(\ell+1)}{2\ell+1} = G.$$

The result follows by shifting the summation index. ∎

Theorem 2.4. Let G denote Catalan's constant. Then, the following identity holds:

$$\sum_{\ell=0}^{\infty} \frac{\mathfrak{B}\left(\ell/2+1\right)}{\ell+1} = 2G - \frac{\pi}{4} \ln(2),$$

Proof. Using (1) and the identity $\frac{1}{\ell+1} = \int_0^1 \tau^\ell d\tau$, the series can be expressed as

$$\sum_{\ell=0}^{\infty} \frac{\mathfrak{B}(\ell/2+1)}{\ell+1} = \sum_{\ell=0}^{\infty} \int_0^1 \int_0^1 \frac{(\tau\sqrt{t})^\ell}{1+t} \, d\tau\, dt.$$

Since the integrand is bounded and continuous on the open unit square, Fubini's theorem justifies interchanging the operations:

$$\begin{aligned}\sum_{\ell=0}^{\infty} \frac{\mathfrak{B}(\ell/2+1)}{\ell+1} &= \int_0^1 \int_0^1 \frac{1}{1+t} \sum_{\ell=0}^{\infty} (\tau\sqrt{t})^\ell \, d\tau\, dt \\ &= \int_0^1 \int_0^1 \frac{1}{(1+t)(1-\tau\sqrt{t})} \, d\tau\, dt.\end{aligned}$$

Consequently,

$$\int_0^1 \frac{1}{1+t} \left[-\frac{\ln(1-\tau\sqrt{t})}{\sqrt{t}} \right]_{\tau=0}^{\tau=1} dt = -\int_0^1 \frac{\ln(1-\sqrt{t})}{\sqrt{t}(1+t)} \, dt.$$

Applying the substitution $t = u^2$, where $dt = 2u\,du$, the integral simplifies to

$$-2\int_0^1 \frac{\ln(1-u)}{1+u^2}\,du. \tag{8}$$

According to [10, Entry 4.291(10)], this integral evaluates to

$$\int_0^1 \frac{\ln(1-u)}{1+u^2}\,du = \frac{\pi}{8}\ln(2) - G. \tag{9}$$

Substituting (9) into (8), we find

$$\sum_{\ell=0}^{\infty} \frac{\mathfrak{B}(\ell/2+1)}{\ell+1} = -2\left(\frac{\pi}{8}\ln(2) - G\right) = 2G - \frac{\pi}{4}\ln(2).$$

□

Theorem 2.5. The following equality holds:

$$\sum_{\ell=0}^{\infty}(-1)^{\ell}\frac{\mathfrak{B}\,(\ell/2+1)}{\ell+1} = \frac{\pi}{4}\ln(2).$$

Proof. Using (1) and the identity $\frac{1}{\ell+1} = \int_0^1 \tau^{\ell} d\tau$, the series becomes

$$\sum_{\ell=0}^{\infty}(-1)^{\ell}\frac{\mathfrak{B}(\ell/2+1)}{\ell+1} = \sum_{\ell=0}^{\infty}(-1)^{\ell}\int_0^1\int_0^1 \frac{(\tau\sqrt{t})^{\ell}}{1+t}\,d\tau\,dt.$$

Since the integrand is bounded and continuous on the open unit square, Fubini's theorem justifies interchanging the operations:

$$\begin{aligned}\sum_{\ell=0}^{\infty}\frac{\mathfrak{B}(\ell/2+1)}{\ell+1} &= \int_0^1\int_0^1 \frac{1}{1+t}\sum_{\ell=0}^{\infty}(-\tau\sqrt{t})^{\ell}\,d\tau\,dt \\ &= \int_0^1\int_0^1 \frac{1}{(1+t)(1+\tau\sqrt{t})}\,d\tau\,dt.\end{aligned}$$

Consequently,

$$\int_0^1 \frac{1}{1+t}\left[\frac{\ln(1+\tau\sqrt{t})}{\sqrt{t}}\right]_{\tau=0}^{\tau=1} dt = \int_0^1 \frac{\ln(1+\sqrt{t})}{\sqrt{t}(1+t)}\,dt.$$

Applying the substitution $t = u^2$, where $dt = 2u\,du$, the integral simplifies to

$$2\int_0^1 \frac{\ln(1+u)}{1+u^2}\,du. \tag{10}$$

According to [10, Entry 4.291(8)], this integral evaluates to

$$\int_0^1 \frac{\ln(1+u)}{1+u^2}\,du = \frac{\pi}{8}\ln(2). \tag{11}$$

Substituting (11) into (10), we find

$$\sum_{\ell=0}^{\infty}(-1)^{\ell}\frac{\mathfrak{B}(\ell/2+1)}{\ell+1} = 2\left(\frac{\pi}{8}\ln(2)\right) = \frac{\pi}{4}\ln(2).$$

□

To further explore these series, we now examine a geometric parameter and the corresponding sums.

Theorem 2.6. For every $a > 1$, one has

$$\sum_{\ell=1}^{\infty} \frac{\mathfrak{B}(\ell)}{a^{\ell}} = \frac{1}{a+1} \ln\left(\frac{2a}{a-1}\right).$$

Proof. Substituting (1) into the series yields

$$\sum_{\ell=0}^{\infty} \frac{\mathfrak{B}(\ell+1)}{a^{\ell+1}} = \sum_{\ell=0}^{\infty} \frac{1}{a^{\ell+1}} \int_0^1 \frac{t^{\ell}}{1+t} dt.$$

By Fubini's theorem, we transpose the infinite summation and integration to obtain

$$\begin{aligned} \sum_{\ell=0}^{\infty} \frac{\mathfrak{B}(\ell+1)}{a^{\ell+1}} &= \frac{1}{a} \int_0^1 \frac{1}{1+t} \sum_{\ell=0}^{\infty} \left(\frac{t}{a}\right)^{\ell} dt \\ &= \frac{1}{a} \int_0^1 \frac{1}{1+t} \left(\frac{a}{a-t}\right) dt = \int_0^1 \frac{1}{(1+t)(a-t)} dt. \end{aligned}$$

Evaluating the integral yields

$$\begin{aligned} \int_0^1 \frac{dt}{(1+t)(a-t)} &= \frac{1}{a+1} \left[\ln(1+t) - \ln(a-t)\right]_0^1 \\ &= \frac{1}{a+1} \left[(\ln 2 - \ln(a-1)) - (0 - \ln a)\right] \\ &= \frac{1}{a+1} \ln\left(\frac{2a}{a-1}\right). \end{aligned}$$

Thus, we find

$$\sum_{\ell=0}^{\infty} \frac{\mathfrak{B}(\ell+1)}{a^{\ell+1}} = \frac{1}{a+1} \ln\left(\frac{2a}{a-1}\right).$$

Re-indexing the summation yields the result. □

Theorem 2.7. The following equality holds for $a > 1$:

$$\sum_{\ell=1}^{\infty} (-1)^{\ell-1} \frac{\mathfrak{B}(\ell)}{a^{\ell}} = \frac{1}{a-1} \ln\left(\frac{2a}{a+1}\right).$$

Proof. Using (1), the alternating series is

$$\sum_{\ell=1}^{\infty} (-1)^{\ell-1} \frac{\mathfrak{B}(\ell)}{a^{\ell}} = \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell-1}}{a^{\ell}} \int_0^1 \frac{t^{\ell-1}}{1+t} dt.$$

By Fubini's theorem, we transpose the order of integration and summation:

$$\begin{aligned} \sum_{\ell=1}^{\infty} (-1)^{\ell-1} \frac{\mathfrak{B}(\ell)}{a^{\ell}} &= \int_0^1 \frac{1}{1+t} \left(\sum_{\ell=1}^{\infty} \frac{(-t)^{\ell-1}}{a^{\ell}}\right) dt \\ &= \int_0^1 \frac{1}{1+t} \left(\frac{1/a}{1-(-t/a)}\right) dt \\ &= \int_0^1 \frac{1}{1+t} \left(\frac{1}{a+t}\right) dt = \frac{1}{a-1} \ln\left(\frac{2a}{a+1}\right). \end{aligned}$$

□

Theorem 2.8. The following equality holds:

$$\sum_{\ell=1}^{\infty} \frac{\mathfrak{B}(\ell)}{\ell^2} = \frac{13}{8}\zeta(3) - \frac{\pi^2}{6}\ln(2).$$

Proof. By (1) and the identity

$$\frac{1}{\ell^2} = \int_0^1 \int_0^1 (\tau t)^{\ell-1}\, d\tau\, dt,$$

the series becomes

$$\sum_{\ell=1}^{\infty} \frac{\mathfrak{B}(\ell)}{\ell^2} = \sum_{\ell=1}^{\infty} \int_0^1 \int_0^1 \int_0^1 \frac{(\tau ts)^{\ell-1}}{1+s}\, d\tau\, dt\, ds.$$

Since the integrand functions are strictly non-negative over the unit cube $(0,1)^3$, Tonelli's theorem validates term-by-term integration.

$$\begin{aligned}\sum_{\ell=1}^{\infty} \frac{\mathfrak{B}(\ell)}{\ell^2} &= \int_0^1 \int_0^1 \int_0^1 \frac{1}{1+s} \left(\sum_{\ell=1}^{\infty} (\tau st)^{\ell-1} \right) d\tau\, dt\, ds \\ &= \int_0^1 \int_0^1 \int_0^1 \frac{1}{(1+s)(1-xst)}\, d\tau\, dt\, ds.\end{aligned}$$

Evaluating the inner integral with respect to τ yields

$$\int_0^1 \frac{1}{(1+s)(1-\tau st)}\, dx = \left[-\frac{\ln(1-\tau st)}{st(1+s)} \right]_{\tau=0}^{\tau=1} = -\frac{\ln(1-st)}{st(1+s)}.$$

Thus, the triple integral reduces to

$$\int_0^1 \int_0^1 -\frac{\ln(1-st)}{st(1+s)}\, dt\, ds = \int_0^1 \frac{1}{s(1+s)} \left[\int_0^1 -\frac{\ln(1-st)}{t}\, dt \right] ds.$$

Recall the integral representation of the dilogarithm: $\int_0^1 \frac{\ln(1-at)}{t}\, dt = -\operatorname{Li}_2(a)$. The expression simplifies to

$$\int_0^1 \frac{\operatorname{Li}_2(s)}{s(1+s)}\, ds.$$

Applying the partial fraction decomposition $\frac{1}{s(1+s)} = \frac{1}{s} - \frac{1}{1+s}$, we obtain

$$\int_0^1 \frac{\operatorname{Li}_2(s)}{s(1+s)}\, ds = \int_0^1 \frac{\operatorname{Li}_2(s)}{s}\, ds - \int_0^1 \frac{\operatorname{Li}_2(s)}{1+s}\, ds. \tag{12}$$

The first integral is a standard result:

$$\int_0^1 \frac{\operatorname{Li}_2(s)}{s}\, ds = \zeta(3). \tag{13}$$

Evaluating the second integral using by parts: we select $u = \operatorname{Li}_2(s)$ and $dv = \frac{1}{1+s} ds$:

$$\begin{aligned}\int_0^1 \frac{\operatorname{Li}_2(s)}{1+s}\, ds &= [\operatorname{Li}_2(s)\ln(1+s)]_0^1 - \int_0^1 \ln(1+s)\left(-\frac{\ln(1-s)}{s} \right) ds \\ &= \ln(2)\operatorname{Li}_2(1) + \int_0^1 \frac{\ln(1-s)\ln(1+s)}{s}\, ds \\ &= \frac{\pi^2}{6}\ln(2) + \int_0^1 \frac{\ln(1-s)\ln(1+s)}{s}\, ds. \end{aligned} \tag{14}$$

According to [20, Eq. (1.14)], the logarithmic integral is

$$\int_0^1 \frac{\ln(1-s)\ln(1+s)}{s}\,ds = -\frac{5}{8}\zeta(3). \tag{15}$$

Substituting (15) into (14), we find

$$\int_0^1 \frac{\mathrm{Li}_2(s)}{1+s}\,ds = \frac{\pi^2}{6}\ln(2) - \frac{5}{8}\zeta(3). \tag{16}$$

Finally, substituting equations (16) and (13) into (12) gives

$$\int_0^1 \frac{\mathrm{Li}_2(s)}{s(1+s)}\,ds = \zeta(3) - \left(\frac{\pi^2}{6}\ln(2) - \frac{5}{8}\zeta(3)\right) = \frac{13}{8}\zeta(3) - \frac{\pi^2}{6}\ln(2). \tag{17}$$

Thus,

$$\sum_{\ell=1}^{\infty} \frac{\mathfrak{B}(\ell)}{\ell^2} = \frac{13}{8}\zeta(3) - \frac{\pi^2}{6}\ln(2).$$

□

Corollary 2.9. Using and Theorem 2.8, the following equalities hold:

$$\sum_{\ell=1}^{\infty} \frac{\mathfrak{B}(\ell+1)}{\ell^2} = \frac{\pi^2}{6}\ln(2) - \frac{5}{8}\zeta(3),$$
$$\sum_{\ell=1}^{\infty} \frac{\mathfrak{B}(\ell+2)}{\ell^2} = \frac{5}{8}\zeta(3) + \frac{\pi^2}{6}(1-\ln 2) - 1,$$
$$\sum_{\ell=1}^{\infty} \frac{\mathfrak{B}(\ell+3)}{\ell^2} = \frac{5}{8} - \frac{5}{8}\zeta(3) + \frac{\pi^2}{6}\ln 2 - \frac{\pi^2}{12}.$$

Having examined infinite series separately, we now investigate generating functions involving central binomial coefficients and Nielsen's beta function.

3 Infinite series involving binomial coefficients

Theorem 3.1. For $0 < \tau < \frac{1}{4}$, the identity

$$\sum_{\ell=0}^{\infty} \binom{2\ell}{\ell}\mathfrak{B}(k+1)\tau^k = \frac{1}{\sqrt{1+4\tau}}\ln\left(\frac{\sqrt{1+4\tau}+1}{\sqrt{1+4\tau}-1}\cdot\frac{\sqrt{1+4\tau}-\sqrt{1-4\tau}}{\sqrt{1+4\tau}+\sqrt{1-4\tau}}\right) \tag{18}$$

holds.

Proof. Using (1), we have

$$\sum_{\ell=0}^{\infty} \binom{2\ell}{\ell}\mathfrak{B}(\ell+1)\tau^\ell = \sum_{\ell=0}^{\infty} \binom{2\ell}{\ell}\tau^\ell \int_0^1 \frac{t^\ell}{1+t}\,dt.$$

Since the power series converges uniformly for $|4\tau t| < 1$, Fubini's theorem allows us to pull the summation inside the integral:

$$\sum_{\ell=0}^{\infty} \binom{2\ell}{\ell}\mathfrak{B}(\ell+1)\tau^\ell = \int_0^1 \frac{1}{1+t}\sum_{\ell=0}^{\infty} \binom{2\ell}{\ell}(\tau t)^\ell\,dt.$$

From [8, Eq. (1)], we have

$$\sum_{\ell=0}^{\infty} \binom{2\ell}{\ell}(\tau t)^\ell = \frac{1}{\sqrt{1-4\tau t}}. \tag{19}$$

Thus,

$$\sum_{\ell=0}^{\infty}\binom{2\ell}{\ell}\mathfrak{B}(\ell+1)\tau^{\ell}=\int_0^1\frac{dt}{(1+t)\sqrt{1-4\tau t}}.$$

Now we evaluate the integral

$$I(\tau)=\int_0^1\frac{dt}{(1+t)\sqrt{1-4\tau t}}.$$

Let $u=\sqrt{1-4\tau t}$. Squaring both sides gives $u^2=1-4\tau t$, which implies

$$t=\frac{1-u^2}{4\tau},\qquad dt=-\frac{u}{2\tau}\,du.$$

Substituting these into $I(\tau)$, we get

$$I(\tau)=\int_1^{\sqrt{1-4\tau}}\frac{1}{\left(1+\frac{1-u^2}{4\tau}\right)u}\left(-\frac{u}{2\tau}\right)du.$$

Simplifying the term $1+\frac{1-u^2}{4\tau}=\frac{4\tau+1-u^2}{4\tau}$ and canceling u, we obtain

$$I(\tau)=\int_{\sqrt{1-4\tau}}^1\frac{4\tau}{4\tau+1-u^2}\cdot\frac{1}{2\tau}\,du=2\int_{\sqrt{1-4\tau}}^1\frac{1}{(1+4\tau)-u^2}\,du.$$

Evaluating the integral yields

$$\begin{aligned}I(\tau)&=2\left[\frac{1}{2\sqrt{1+4\tau}}\ln\left(\frac{\sqrt{1+4\tau}+u}{\sqrt{1+4\tau}-u}\right)\right]_{\sqrt{1-4\tau}}^{1}\\&=\frac{1}{\sqrt{1+4\tau}}\left[\ln\left(\frac{\sqrt{1+4\tau}+1}{\sqrt{1+4\tau}-1}\right)-\ln\left(\frac{\sqrt{1+4\tau}+\sqrt{1-4\tau}}{\sqrt{1+4\tau}-\sqrt{1-4\tau}}\right)\right].\end{aligned}$$

Combining the logarithms, we arrive at

$$I(\tau)=\frac{1}{\sqrt{1+4\tau}}\ln\left(\frac{(\sqrt{1+4\tau}+1)(\sqrt{1+4\tau}-\sqrt{1-4\tau})}{(\sqrt{1+4\tau}-1)(\sqrt{1+4\tau}+\sqrt{1-4\tau})}\right).$$

□

The following theorem establishes a new generating-function for the Nielsen $\mathfrak{B}$ function.

Theorem 3.3. For $0<\tau<\frac{1}{4}$, we have

$$\sum_{\ell=1}^{\infty}\ell\binom{2\ell}{\ell}\mathfrak{B}(\ell+1)\tau^{\ell-1}=\frac{1-\sqrt{1-4\tau}}{\tau(1+4\tau)\sqrt{1-4\tau}}-\frac{2}{(1+4\tau)^{3/2}}\ln\left(\frac{\sqrt{1+4\tau}+1}{\sqrt{1+4\tau}-1}\cdot\frac{\sqrt{1+4\tau}-\sqrt{1-4\tau}}{\sqrt{1+4\tau}+\sqrt{1-4\tau}}\right).$$

Proof. The desired result follows from differentiating (18) with respect to τ. □

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Theorem 3.5. The following equality holds:

$$\sum_{\ell=0}^{\infty}\binom{2\ell}{\ell}\frac{\mathfrak{B}(\ell+1)}{4^{\ell+1}(\ell+1)}=\left(\frac{\sqrt{2}-3}{2}\right)\ln(2)+\sqrt{2}\ln(1+\sqrt{2}).$$

Proof. Upon setting $\tau=4u$ in the integral representation of Catalan numbers (see [7, Eq. (10)])

$$c_n=\frac{1}{n+1}\binom{2n}{n}=\frac{1}{2\pi}\int_0^4\tau^n\sqrt{\frac{4-\tau}{\tau}}\,d\tau,$$

we obtain

$$c_n = \frac{1}{n+1}\binom{2n}{n} = \frac{4^{n+1}}{2\pi}\int_0^1 u^n\sqrt{\frac{1-u}{u}}\,du.$$

Using this representation, we have

$$\begin{aligned}\sum_{\ell=0}^{\infty}\binom{2\ell}{\ell}\frac{\mathfrak{B}(\ell+1)}{4^{\ell+1}(\ell+1)} &= \frac{1}{2\pi}\sum_{\ell=0}^{\infty}\int_0^1 u^\ell\sqrt{\frac{1-u}{u}}\,du\int_0^1\frac{t^\ell}{1+t}\,dt\\ &= \frac{1}{2\pi}\int_0^1\int_0^1\frac{1}{1+t}\sqrt{\frac{1-u}{u}}\sum_{\ell=0}^{\infty}(ut)^\ell\,dt\,du\\ &= \frac{1}{2\pi}\int_0^1\int_0^1\sqrt{\frac{1-u}{u}}\frac{1}{(1+t)(1-ut)}\,dt\,du. \end{aligned}\tag{20}$$

A straightforward calculation shows that the inner integral evaluates to

$$\int_0^1\frac{dt}{(1+t)(1-ut)} = \frac{1}{1+u}\big[\ln(2)-\ln(1-u)\big].\tag{21}$$

Substituting (21) into (20), we obtain

$$\frac{1}{2\pi}\int_0^1\frac{1}{1+u}\sqrt{\frac{1-u}{u}}\big[\ln(2)-\ln(1-u)\big]\,du.$$

Define

$$I_1 = \frac{\ln(2)}{2\pi}\int_0^1\sqrt{\frac{1-u}{u}}\frac{du}{1+u}, \qquad I_2 = \frac{1}{2\pi}\int_0^1\sqrt{\frac{1-u}{u}}\frac{\ln(1-u)}{1+u}\,du.$$

Using the substitution $u=\sin^2\theta$, the first integral I_1 becomes

$$I_1 = \frac{\ln(2)}{2\pi}\int_0^{\pi/2}\sqrt{\frac{1-\sin^2\theta}{\sin^2\theta}}\frac{\sin(2\theta)}{1+\sin^2\theta}\,d\theta = \frac{\ln(2)}{\pi}\int_0^{\pi/2}\frac{\cos^2\theta}{1+\sin^2\theta}\,d\theta.\tag{22}$$

A straightforward calculation yields

$$I_1 = \frac{\ln(2)}{\pi}\cdot\frac{\pi}{2}(\sqrt{2}-1) = \frac{\ln(2)}{2}(\sqrt{2}-1).\tag{23}$$

For the second integral I_2, using $u=\sin^2\theta$, we have

$$\begin{aligned}I_2 &= \frac{1}{2\pi}\int_0^{\pi/2}\sqrt{\frac{\cos^2\theta}{\sin^2\theta}}\frac{\ln(\cos^2\theta)}{1+\sin^2\theta}\,2\sin\theta\cos\theta\,d\theta\\ &= \frac{2}{\pi}\int_0^{\pi/2}\frac{\cos^2\theta}{1+\sin^2\theta}\ln(\cos\theta)\,d\theta.\end{aligned}$$

Now apply the substitution $t=\tan\theta$. Then

$$\cos^2\theta = \frac{1}{1+t^2}, \quad d\theta = \frac{dt}{1+t^2}, \quad \ln(\cos\theta) = -\frac{1}{2}\ln(1+t^2).$$

Therefore, I_2 reduces to

$$\begin{aligned}I_2 &= -\frac{1}{\pi}\int_0^{\infty}\frac{\ln(1+t^2)}{(1+t^2)(1+2t^2)}\,dt\\ &= \frac{1}{\pi}\int_0^{\infty}\frac{\ln(1+t^2)}{1+t^2}\,dt - \frac{2}{\pi}\int_0^{\infty}\frac{\ln(1+t^2)}{1+2t^2}\,dt.\end{aligned}$$

Using the standard integral (see [10, Entry 4.295(22)])

$$\int_0^\infty \frac{\ln(1+p^2t^2)}{r^2+q^2t^2}\,dt = \frac{\pi}{qr}\ln\left(\frac{q+pr}{r}\right),$$

we find

$$\begin{aligned} I_2 &= \frac{1}{\pi}(\pi\ln 2) - \frac{2}{\pi}\left[\frac{\pi}{\sqrt{2}}\ln(1+\sqrt{2})\right] \\ &= \ln 2 - \sqrt{2}\ln(1+\sqrt{2}). \end{aligned}$$

Therefore, the final result is

$$\begin{aligned} I_1 - I_2 &= \frac{\sqrt{2}-1}{2}\ln 2 - \big(\ln 2 - \sqrt{2}\ln(1+\sqrt{2})\big) \\ &= \left(\frac{\sqrt{2}-3}{2}\right)\ln 2 + \sqrt{2}\ln(1+\sqrt{2}). \end{aligned}$$

□

Theorem 3.6. The following equality holds:

$$\sum_{\ell=0}^{\infty} \frac{\mathfrak{B}(\ell+1)}{(3\ell+1)\binom{3\ell}{\ell}} = \int_0^1 \frac{\ln(2)-\ln(1-\tau+2\tau^2-\tau^3)}{1+\tau-2\tau^2+\tau^3}\,d\tau. \tag{24}$$

Proof. Utilizing the properties of the beta function, we have

$$\frac{1}{(3\ell+1)\binom{3\ell}{\ell}} = \mathrm{B}(\ell+1, 2\ell+1) = \int_0^1 \tau^\ell(1-\tau)^{2\ell}\,dx. \tag{25}$$

Thus, the series can be expressed as

$$\begin{aligned} \sum_{\ell=0}^{\infty} \frac{\mathfrak{B}(\ell+1)}{(3\ell+1)\binom{3\ell}{\ell}} &= \sum_{\ell=0}^{\infty}\int_0^1 \tau^\ell(1-\tau)^{2\ell}\,d\tau \int_0^1 \frac{t^\ell}{1+t}\,dt \\ &= \int_0^1\int_0^1 \frac{1}{1+t}\sum_{\ell=0}^{\infty}\big(\tau t(1-\tau)^2\big)^\ell\,dt\,d\tau \\ &= \int_0^1\int_0^1 \frac{1}{(1+t)\big(1-\tau t(1-\tau)^2\big)}\,dt\,d\tau. \end{aligned}$$

Set $a = \tau(1-\tau)^2$. By partial fraction decomposition, the inner integral evaluates to

$$\int_0^1 \frac{dt}{(1+t)(1-at)} = \frac{1}{1+a}\big(\ln(2)-\ln(1-a)\big).$$

Substituting $a = \tau(1-\tau)^2 = \tau - 2\tau^2 + \tau^3$, we obtain

$$\int_0^1 \frac{dt}{(1+t)\big(1-\tau t(1-\tau)^2\big)} = \frac{\ln(2)-\ln(1-\tau+2\tau^2-\tau^3)}{1+\tau-2\tau^2+\tau^3}.$$

Therefore, the outer integral becomes

$$\int_0^1 \frac{\ln(2)-\ln(1-\tau+2\tau^2-\tau^3)}{1+\tau-2\tau^2+\tau^3}\,d\tau.$$

Hence,

$$\sum_{k=0}^{\infty} \frac{\mathfrak{B}(\ell+1)}{(3\ell+1)\binom{3\ell}{\ell}} = \int_0^1 \frac{\ln(2)-\ln(1-\tau+2\tau^2-\tau^3)}{1+\tau-2\tau^2+\tau^3}\,d\tau.$$

□

4 Discussion

The results obtained in this paper demonstrate the effectiveness of integral representations in the evaluation of infinite series involving Nielsen's beta function and central binomial coefficients. By transforming the series into suitable integral forms, we derive several closed-form identities and obtain new representations for classical constants such as $\zeta(3)$, Catalan's constant (G), and $\ln(2)$. A key feature of our approach is the interchange of summation and integration, which is justified whenever necessary by Tonelli's and Fubini's theorems.In some cases, they yield closed-form expressions, while in others they lead to more challenging integral representations whose evaluation remains an interesting direction for future research. Overall, these results further highlight the close connections between Nielsen's beta function, binomial coefficient series, special functions, and classical mathematical constants.

5 Conclusion and Open Problem

5.1 Conclusion

Throughout this work, we have established unique evaluations for infinite sums involving central binomial terms and the Nielsen beta function. By employing integral representations and Fubini-type arguments, we reduced these series to logarithmic integrals and explicit evaluations in terms of classical constants such as $\zeta(3)$, Catalan's constant, and $\ln(2)$. These results extend the existing literature on series involving harmonic numbers and binomial coefficients by incorporating the Nielsen beta function, thereby providing new analytical insights and representations.

5.2 Open problem

Determine a closed-form formula for the remaining integral from Theorem 3.6:

$$\int_0^1 \frac{\ln(2) - \ln(1 - \tau + 2\tau^2 - \tau^3)}{1 + \tau - 2\tau^2 + \tau^3}\, d\tau.$$

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