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Deferred Cesàro Convergence of Bicomplex Sequences Using D-Orlicz Function

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Abstract: *In this paper, we introduce strong deferred Cesàro null, strong deferred Cesàro convergent and strong deferred Cesàro bounded bicomplex sequence spaces using D-Orlicz function with respect to the hyperbolic norm (D-norm). We also study some algebraic, geometric and topological properties of these spaces and establish some inclusion relations among the spaces.*

Keywords: D-Orlicz function, D-norm, Bicomplex numbers, Deferred Cesàro convergence.

1 Introduction

The credit of introducing bicomplex numbers goes to Segre [15]. Later on, many researchers worked on the bicomplex sequences and their convergence, some of them are Sager and Sağır [14], Değirmen and Sağır [6], Bera and Tripathy [4] and many others. Researchers such as Cockle [5], Lie and Scheffers [11], Rochon and Shapiro [13], Kumar, Sharma, Tundup and Wazir [10], Bera and Tripathy [3], Bera, Tamuli and Tripathy [2] studied hyperbolic numbers and sequence spaces using D-Orlicz function with respect to the hyperbolic norm.

The concept of deferred Cesàro mean was introduced by Agnew [1]. After that, many mathematicians studied deferred statistical convergence. For such studies, we refer to the works of Küçükaslan and Yılmaztürk [9], Şengül, Et and Işık [16], Et, Cinar and Kandemir [7] etc.

The main aim of this paper is to introduce different types of strong deferred Cesàro sequence spaces using D-Orlicz function with respect to the D-norm and study their different properties. Throughout the paper, $\mathbb{C}_0$, $\mathbb{C}_1$ and $\mathbb{C}_2$ denote the sets of real numbers, complex numbers and bicomplex numbers, respectively.

Definition 1. *[8] A convex, continuous and non-decreasing function* $M : [0, \infty) \to [0, \infty)$ *with conditions* $M(0) = 0,\ M(x) > 0$ *if* $x > 0$ *and* $M(x) \to \infty$ *as* $x \to \infty$ *is termed an Orlicz function.*
The Lindenstrauss and Tzafriri [12] constructed the Orlicz space

$$\ell_M = \left\{ x \in \omega : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty,\ \text{for some } \rho > 0 \right\}$$

and showed that ℓ_M *is a Banach space with norm*

$$\|x\| = \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}.$$

2 Definitions and Preliminaries

Bicomplex Numbers: [15]
A bicomplex number can be written in the form

$$\eta = d_1 + i_1 d_2 + i_2 d_3 + i_1 i_2 d_4 = z_1 + i_2 z_2,$$

where $d_1, d_2, d_3, d_4 \in \mathbb{C}_0$, $z_1, z_2 \in \mathbb{C}_1$, and i_1, i_2 are independent imaginary units. The unique idempotent representation is

$$\eta = \mu_1 e_1 + \mu_2 e_2,$$

where

$$\mu_1 = z_1 - i_1 z_2, \qquad \mu_2 = z_1 + i_1 z_2.$$

The non-trivial idempotent elements

$$e_1 = \frac{1 + i_1 i_2}{2} \quad \text{and} \quad e_2 = \frac{1 - i_1 i_2}{2}$$

satisfy the relations

$$e_1^2 = e_1, \qquad e_2^2 = e_2, \qquad e_1 e_2 = 0, \quad \text{and} \quad e_1 + e_2 = 1.$$

The Euclidean norm on $\mathbb{C}_2$ is

$$\|\eta\|_{\mathbb{C}_2} = \sqrt{d_1^2 + d_2^2 + d_3^2 + d_4^2} = \sqrt{|z_1|^2 + |z_2|^2} = \sqrt{\frac{|\mu_1|^2 + |\mu_2|^2}{2}}.$$

With respect to the Euclidean norm, $\mathbb{C}_2$ is a Banach space.
If $\left|z_1^2 + z_2^2\right| = 0$, then the bicomplex number $\eta = z_1 + i_2 z_2$ is called singular. The set of all singular bicomplex numbers is denoted by $\mathbb{O}_2$.

Hyperbolic Numbers: [5]
A bicomplex number in the form

$$\beta = d_1 + i_1 i_2 d_2, \qquad \text{where } d_1, d_2 \in \mathbb{C}_0,$$

is called a hyperbolic number. Its idempotent representation is

$$\beta = x_1 e_1 + x_2 e_2,$$

where

$$x_1 = d_2 + d_1, \qquad x_2 = d_2 - d_1.$$

The set of hyperbolic numbers is denoted by D. The set of positive hyperbolic numbers is

$$D^+ = \{\, x_1 e_1 + x_2 e_2 : x_1, x_2 \geq 0 \,\}.$$

The hyperbolic norm (D-norm) on $\mathbb{C}_2$ is defined by

$$|\eta|_D = |\mu_1| e_1 + |\mu_2| e_2 \in D_+, \quad \text{where } \eta \in \mathbb{C}_2.$$

The partial order relation in D is given by

$$\beta \leq' \gamma \iff \gamma - \beta \in D_+, \quad \text{for all } \beta, \gamma \in D.$$

We have

$$|\eta + \xi|_D \leq' |\eta|_D + |\xi|_D \text{ and } |\eta \xi|_D = |\eta|_D |\xi|_D, \quad \text{for all } \eta, \xi \in \mathbb{C}_2.$$

Let $A \subset D$, $A_1 = \{x_1 : x_1 e_1 + x_2 e_2 \in A\}$ *and* $A_2 = \{x_2 : x_1 e_1 + x_2 e_2 \in A\}$.
Then, $\sup\limits_D A = e_1 \sup A_1 + e_2 \sup A_2$ and $\inf\limits_D A = e_1 \inf A_1 + e_2 \inf A_2$.
The set of extended non-negative hyperbolic numbers is given by

$$D_+^* = \{\mu_1 e_1 + \mu_2 e_2 : \mu_1, \mu_2 > 0\} \cup \{\infty\} \cup \{-\infty\} \cup \{\infty e_1 + \mu_2 e_2\} \cup \{\mu_1 e_1 - \infty e_2\}.$$

Definition 2. *[17] A sequence space X is called solid (or normal) if whenever $(x_k) \in X$ and (a_k) is any scalar sequence satisfying $|a_k| \leq 1$ for every $k \in \mathbb{N}$, then $(a_k x_k) \in X$.*

Definition 3. *[17] A sequence space X is called convergence free if for any $(x_k) \in X$, the condition $x_k = 0$ implies $y_k = 0$, then $(y_k) \in X$.*

Definition 4. *[10] A D-valued convex function is a function $\Upsilon_D : D \to D_+^*$ satisfying the condition*

$$\Upsilon_D(\alpha\eta + (1-\alpha)\xi) \leq' \alpha\Upsilon_D(\eta) + (1-\alpha)\Upsilon_D(\xi),$$

for all $\eta, \xi \in D$ and $0 \leq' \alpha \leq' 1$.

Definition 5. *[10] A D-Orlicz function is a convex function $\Upsilon_D : D_+ \to D_+^*$ having the following properties:*
(i) $\Upsilon_D(0_D) = 0_D$;
(ii) $\lim_{\eta\to\infty} \Upsilon_D(\eta) = \infty^*$, where $\infty^* = \mu_1 e_1 + \infty e_2 = \infty e_1 + \mu_2 e_2 = \infty e_1 + \infty e_2$ and $\lim_{\eta\to\infty} \Upsilon_D(\eta)$ must exist along every line in the hyperbolic plane, and all such limits must be equal.
A D-Orlicz function is denoted by M_D.

Definition 6. *[3] A D-Orlicz function M_D is said to satisfy Δ_D^2-condition if there exists a hyperbolic number $K \geq' 0$ and η_0 (depending upon K) such that*

$$M_D((2e_1 + 2e_2)\eta) \leq' KM_D(\eta), \quad \textit{for all } 0 \leq' \eta \leq' \eta_0.$$

Definition 7. *[3] A D-paranorm is a function $g : \mathbb{C}_2 \to D_+^*$ having the following properties:*
$DP_1 :\quad g(\eta) \geq' 0_D, \quad$ *for all* $\eta \in \mathbb{C}_2$;
$DP_2 :\quad g(-\eta) = g(\eta), \quad$ *for all* $\eta \in \mathbb{C}_2$;
$DP_3 :\quad g(\eta + \xi) \leq' g(\eta) + g(\xi), \quad$ *for all* $\eta, \xi \in \mathbb{C}_2$;
$DP_4 :\quad \alpha_k \to \alpha,\ |\eta_k - \ell|_D \to 0_D \Rightarrow |\alpha_k\eta_k - \alpha\ell|_D \to 0_D.$

Definition 8. *[9] The deferred Cesàro mean of a real sequence $x = (x_n)$ is defined by*

$$\left(D^{\rho,\sigma}(x)\right)_n = \frac{1}{\sigma_n - \rho_n} \sum_{k=\rho_n+1}^{\sigma_n} x_k, \qquad n \in \mathbb{N},$$

where

$$\rho = \{\rho_n : n \in \mathbb{N}\} \textit{ and } \sigma = \{\sigma_n : n \in \mathbb{N}\}$$

are non-negative integer sequences with conditions

$$\rho_n < \sigma_n \quad \textit{and} \quad \lim_{n\to\infty} \sigma_n = \infty \tag{2.1}$$

Definition 9. *[9] The real sequence $x = (x_k)$ is termed strongly deferred Cesàro convergent to $t \in \mathbb{C}_0$ if*

$$\lim_{n\to\infty} \frac{1}{\sigma_n - \rho_n} \sum_{k=\rho_n+1}^{\sigma_n} |x_k - t| = 0.$$

3 Main Results

Throughout this section, the sequences of non-negative integers $\rho = \{\rho_n : n \in \mathbb{N}\}$ and $\sigma = \{\sigma_n : n \in \mathbb{N}\}$ satisfying (2.1) are used and ω^* denotes the space of all bicomplex sequences.

In this section, we introduce different types of strong deferred Cesàro convergent sequence spaces of bicomplex numbers using D-Orlicz function with respect to the D-norm and study their different properties. We now define the following sets:

$$[D_c^*, M_D] = \left\{\eta \in \omega^* : \lim_{n\to\infty} \frac{1}{\sigma_n - \rho_n} \sum_{k=\rho_n+1}^{\sigma_n} M_D\left(\frac{|\eta_k - \ell|_D}{\beta}\right) = 0_D, \text{ for some } \ell \in \mathbb{C}_2 \text{ and hyperbolic number } \beta >' 0\right\};$$

$$[D_0^*, M_D] = \left\{\eta \in \omega^* : \lim_{n\to\infty} \frac{1}{\sigma_n - \rho_n} \sum_{k=\rho_n+1}^{\sigma_n} M_D\left(\frac{|\eta_k|_D}{\beta}\right) = 0_D, \text{ for some hyperbolic number } \beta >' 0\right\};$$

$$[D_\infty^*, M_D] = \left\{\eta \in \omega^* : \lim_{n\to\infty} \frac{1}{\sigma_n - \rho_n} \sum_{k=\rho_n+1}^{\sigma_n} M_D\left(\frac{|\eta_k|_D}{\beta}\right) <' \infty, \text{ for some hyperbolic number } \beta >' 0\right\}.$$

Theorem 3.1. *The sets* $[D_c^*, M_D]$, $[D_0^*, M_D]$ *and* $[D_\infty^*, M_D]$ *are linear spaces over* $\mathbb{C}_2 \setminus \mathbb{O}_2$.

Proof. Suppose $\eta = (\eta_k), \xi = (\xi_k) \in [D_0^*, M_D]$. Then there exist hyperbolic numbers β_1, $\beta_2 >' 0$ such that

$$\lim_{n\to\infty} \frac{1}{\sigma_n - \rho_n} \sum_{k=\rho_n+1}^{\sigma_n} M_D\left(\frac{|\eta_k|_D}{\beta_1}\right) = 0_D$$

and

$$\lim_{n\to\infty} \frac{1}{\sigma_n - \rho_n} \sum_{k=\rho_n+1}^{\sigma_n} M_D\left(\frac{|\xi_k|_D}{\beta_2}\right) = 0_D.$$

Let $c_1, c_2 \in \mathbb{C}_2 \setminus \mathbb{O}_2$ and $\beta = \max\{2|c_1|_D\beta_1,\, 2|c_2|_D\beta_2\}$.
Since M_D is non-decreasing and D-convex, we have

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{\sigma_n - \rho_n} \sum_{k=\rho_n+1}^{\sigma_n} M_D\left(\frac{|c_1\eta_k + c_2\xi_k|_D}{\beta}\right)\\
&\leq' \lim_{n\to\infty} \frac{1}{\sigma_n - \rho_n} \sum_{k=\rho_n+1}^{\sigma_n} M_D\left(\frac{|c_1|_D|\eta_k|_D + |c_2|_D|\xi_k|_D}{\beta}\right)\\
&\leq' \lim_{n\to\infty} \frac{1}{\sigma_n - \rho_n} \sum_{k=\rho_n+1}^{\sigma_n} M_D\left(\frac{|\eta_k|_D}{\beta_1}\right) + \lim_{n\to\infty} \frac{1}{\sigma_n - \rho_n} \sum_{k=\rho_n+1}^{\sigma_n} M_D\left(\frac{|\xi_k|_D}{\beta_2}\right)\\
&= 0_D.
\end{aligned}$$

Hence $[D_0^*, M_D]$ is a linear space over $\mathbb{C}_2 \setminus \mathbb{O}_2$. The other cases can be handled similarly. □

Theorem 3.2. $[D_0^*, M_D] \subset [D_c^*, M_D] \subset [D_\infty^*, M_D]$, *the inclusions are proper.*

Proof. The inclusion $[D_0^*, M_D] \subset [D_c^*, M_D]$ is obvious. Now, let $\eta = (\eta_k) \in [D_c^*, M_D]$. Then there exists a hyperbolic number $\beta >' 0$ such that

$$\lim_{n\to\infty} \frac{1}{\sigma_n - \rho_n} \sum_{k=\rho_n+1}^{\sigma_n} M_D\left(\frac{|\eta_k - \ell|_D}{\beta}\right) = 0_D.$$

Now,

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{\sigma_n - \rho_n} \sum_{k=\rho_n+1}^{\sigma_n} M_D\left(\frac{|\eta_k|_D}{2\beta}\right)\\
&= \lim_{n\to\infty} \frac{1}{\sigma_n - \rho_n} \sum_{k=\rho_n+1}^{\sigma_n} M_D\left(\frac{|(\eta_k - \ell) + \ell|_D}{2\beta}\right)\\
&\leq' \lim_{n\to\infty} \frac{1}{\sigma_n - \rho_n} \sum_{k=\rho_n+1}^{\sigma_n} M_D\left(\frac{|\eta_k - \ell|_D + |\ell|_D}{2\beta}\right)\\
&\leq' \frac{1}{2} \cdot \lim_{n\to\infty} \frac{1}{\sigma_n - \rho_n} \sum_{k=\rho_n+1}^{\sigma_n} M_D\left(\frac{|\eta_k - \ell|_D}{\beta}\right) + \frac{1}{2} \cdot \lim_{n\to\infty} \frac{1}{\sigma_n - \rho_n} \sum_{k=\rho_n+1}^{\sigma_n} M_D\left(\frac{|\ell|_D}{\beta}\right)\\
&<' \infty.
\end{aligned}$$

Hence, $\eta \in [D_\infty^*, M_D]$. So, $[D_c^*, M_D] \subset [D_\infty^*, M_D]$.

To show inclusions are proper, the following examples are used:
let $M_D(t) = t^2$, for all $t \in D_+$, $\rho_n = 0$ and $\sigma_n = n$ for all $n \in \mathbb{N}$.
Consider a sequence $\eta = (\eta_k)$ in $\mathbb{C}_2$ defined by

$$\eta_k = e_1 + e_2, \text{ for all } k \in \mathbb{N}.$$

Then $\eta \in [D_c^*, M_D]$, but $\eta \notin [D_0^*, M_D]$.
Consider another sequence $\xi = (\xi_k)$ in $\mathbb{C}_2$ defined by

$$\xi_k = \begin{cases} e_1 + e_2, & \text{if } k \text{ is even}, \\ 2(e_1 + e_2), & \text{otherwise}. \end{cases}$$

Then, $\xi \in [D_\infty^*, M_D],$ *but* $\xi \notin [D_c^*, M_D]$. □

Theorem 3.3. *The spaces* $[D_0^*, M_D]$ *and* $[D_\infty^*, M_D]$ *are* $\mathbb{C}_2$*-solid.*

Proof. Suppose $\eta = (\eta_k) \in [D_0^*, M_D]$. Then there exist a hyperbolic number $\beta >' 0$ such that

$$\lim_{n\to\infty} \frac{1}{\sigma_n - \rho_n} \sum_{k=\rho_n+1}^{\sigma_n} M_D\left(\frac{|\eta_k|_D}{\beta}\right) = 0_D.$$

Let a bicomplex scalar sequence (α_k) satisfying $|\alpha_k|_D \leq' 1$, for all $k \in \mathbb{N}$.
Then

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{\sigma_n - \rho_n} \sum_{k=\rho_n+1}^{\sigma_n} M_D\left(\frac{|\alpha_k \eta_k|_D}{\beta}\right) \\ &= \lim_{n\to\infty} \frac{1}{\sigma_n - \rho_n} \sum_{k=\rho_n+1}^{\sigma_n} M_D\left(\frac{|\alpha_k|_D |\eta_k|_D}{\beta}\right) \\ &\leq' \lim_{n\to\infty} \frac{1}{\sigma_n - \rho_n} \sum_{k=\rho_n+1}^{\sigma_n} M_D\left(\frac{|\eta_k|_D}{\beta}\right) \\ &= 0_D. \end{aligned}$$

Hence $(\alpha_k \eta_k) \in [D_0^*, M_D]$. Hence $[D_0^*, M_D]$ is solid. Similarly, the space $[D_\infty^*, M_D]$ is solid. □

The spaces $[D_0^*, M_D]$, $[D_c^*, M_D]$ and $[D_\infty^*, M_D]$ are not convergence free.
It follows from the following example:
Let $M_D(t) = t^2$ for all $t \in D_+$, $\rho_n = 0$ and $\sigma_n = n$, for all $n \in \mathbb{N}$. Consider two sequences in $\mathbb{C}_2$ defined by

$$\eta_k = \frac{1}{k}(e_1 + e_2),$$

and

$$\xi_k = k(e_1 + e_2), \qquad \text{for all } k \in \mathbb{N}.$$

Then $(\eta_k) \in [D_p^*, M_D]$, but $(\xi_k) \notin [D_p^*, M_D],$ *where* $p = 0,\ c,\ \infty$.

Theorem 3.4. *The space* $[D_\infty^*, M_D]$ *is D-convex.*

Proof. Suppose $\eta, \xi \in [D_\infty^*, M_D]$ and $0 \leq' \alpha \leq' 1$. Then

$$\lim_{n\to\infty} \frac{1}{\sigma_n - \rho_n} \sum_{k=\rho_n+1}^{\sigma_n} M_D\left(\frac{|\eta_k|_D}{\beta_1}\right) <' \infty$$

and

$$\lim_{n\to\infty} \frac{1}{\sigma_n - \rho_n} \sum_{k=\rho_n+1}^{\sigma_n} M_D\left(\frac{|\xi_k|_D}{\beta_2}\right) <' \infty,$$

for some hyperbolic numbers $\beta_1, \beta_2 >' 0$.
Put $\beta = \max\{\beta_1, \beta_2\}$. Then

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{\sigma_n - \rho_n} \sum_{k=\rho_n+1}^{\sigma_n} M_D\left(\frac{|\alpha\eta_k + (1-\alpha)\xi_k|_D}{\beta}\right)\\
&\leq' \lim_{n\to\infty} \frac{1}{\sigma_n - \rho_n} \sum_{k=\rho_n+1}^{\sigma_n} M_D\left(\frac{\alpha|\eta_k|_D}{\beta} + \frac{(1-\alpha)|\xi_k|_D}{\beta}\right)\\
&\leq' \alpha \lim_{n\to\infty} \frac{1}{\sigma_n - \rho_n} \sum_{k=\rho_n+1}^{\sigma_n} M_D\left(\frac{|\eta_k|_D}{\beta_1}\right) + (1-\alpha) \lim_{n\to\infty} \frac{1}{\sigma_n - \rho_n} \sum_{k=\rho_n+1}^{\sigma_n} M_D\left(\frac{|\xi_k|_D}{\beta_2}\right)\\
&<' \infty.
\end{aligned}$$

Hence $(\alpha\eta + (1-\alpha)\xi) \in [D^*_\infty, M_D]$, and so it is D-convex. □

Theorem 3.5. *If M^1_D and M^2_D are two D-Orlicz functions satisfying Δ^2_D-condition, then*

$$\left[D^*_p, M^1_D\right] \cap \left[D^*_p, M^2_D\right] \subset \left[D^*_p, M^1_D + M^2_D\right],$$

where $p = 0,\ c,\ \infty$.

Proof. Suppose $\eta = (\eta_k) \in \left[D^*_0, M^1_D\right] \cap \left[D^*_0, M^2_D\right]$. Then there exist hyperbolic numbers $\beta_1, \beta_2 >' 0$ such that

$$\lim_{n\to\infty} \frac{1}{\sigma_n - \rho_n} \sum_{k=\rho_n+1}^{\sigma_n} M^1_D\left(\frac{|\eta_k|_D}{\beta_1}\right) = 0_D$$

and

$$\lim_{n\to\infty} \frac{1}{\sigma_n - \rho_n} \sum_{k=\rho_n+1}^{\sigma_n} M^2_D\left(\frac{|\eta_k|_D}{\beta_2}\right) = 0_D.$$

Put $\beta = \max\{\beta_1, \beta_2\}$. Then

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{\sigma_n - \rho_n} \sum_{k=\rho_n+1}^{\sigma_n} [M^1_D + M^2_D]\left(\frac{|\eta_k|_D}{\beta}\right)\\
&= \lim_{n\to\infty} \frac{1}{\sigma_n - \rho_n} \sum_{k=\rho_n+1}^{\sigma_n} M^1_D\left(\frac{|\eta_k|_D}{\beta}\right) + \lim_{n\to\infty} \frac{1}{\sigma_n - \rho_n} \sum_{k=\rho_n+1}^{\sigma_n} M^2_D\left(\frac{|\eta_k|_D}{\beta}\right)\\
&\leq' \lim_{n\to\infty} \frac{1}{\sigma_n - \rho_n} \sum_{k=\rho_n+1}^{\sigma_n} M^1_D\left(\frac{|\eta_k|_D}{\beta_1}\right) + \lim_{n\to\infty} \frac{1}{\sigma_n - \rho_n} \sum_{k=\rho_n+1}^{\sigma_n} M^2_D\left(\frac{|\eta_k|_D}{\beta_2}\right)\\
&= 0_D.
\end{aligned}$$

Hence the result. The other cases are similarly be handled. □

Theorem 3.6. *If M^1_D and M^2_D are two D-Orlicz functions satisfying Δ^2_D-condition, then*

$$\left[D^*_\infty, M^2_D\right] \subset \left[D^*_\infty, M^1_D * M^2_D\right].$$

Proof. Let $\eta = (\eta_k) \in \left[D^*_\infty, M^2_D\right]$. Then

$$\lim_{n\to\infty} \frac{1}{\sigma_n - \rho_n} \sum_{k=\rho_n+1}^{\sigma_n} M^2_D\left(\frac{|\eta_k|_D}{\beta}\right) <' \infty,$$

for some hyperbolic number $\beta >' 0$. Let

$$u = M_D^2\left(\frac{|\eta_k|_D}{\beta}\right).$$

Since M_D^1 satisfies Δ_D^2-condition, there exists a hyperbolic number $K \geq' 0$ and u_0 (depending upon K) such that

$$M_D^1(u) \leq' KuM_D^1(2e_1+2e_2),$$

for all $0 \leq' u \leq' u_0$.
Now,

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1}{\sigma_n-\rho_n}\sum_{k=\rho_n+1}^{\sigma_n}(M_D^1 * M_D^2)\left(\frac{|\eta_k|_D}{\beta}\right)\\
&= \lim_{n\to\infty}\frac{1}{\sigma_n-\rho_n}\sum_{k=\rho_n+1}^{\sigma_n}M_D^1\left(M_D^2\left(\frac{|\eta_k|_D}{\beta}\right)\right)\\
&= \lim_{n\to\infty}\frac{1}{\sigma_n-\rho_n}\sum_{k=\rho_n+1}^{\sigma_n}M_D^1(u)\\
&\leq' \lim_{n\to\infty}\frac{1}{\sigma_n-\rho_n}\sum_{k=\rho_n+1}^{\sigma_n}KuM_D^1(2e_1+2e_2)\\
&<' \infty.
\end{aligned}$$

Hence, $\eta \in \left[D_\infty^*, M_D^1 * M_D^2\right]$. □

Theorem 3.7. *Suppose M_D is a D-Orlicz function. Then the space $[D_\infty^*, M_D]$ is a D-paranorm space with*

$$g(\eta) = \inf\left\{\beta : \frac{1}{\sigma_n-\rho_n}\sum_{k=\rho_n+1}^{\sigma_n}\left[M_D\left(\frac{|\eta_k|_D}{\beta}\right)\right] \leq' 1,\ \textit{for some hyperbolic number } \beta >' 0\right\}.$$

Proof. As $\beta >' 0$, we have $g(\eta) >' 0$ and $g(-\eta) = g(\eta)$, for all $\eta \in [D_\infty^*, M_D]$.
Next, let $\eta, \xi \in [D_\infty^*, M_D]$. Then there exist hyperbolic numbers $\beta_1, \beta_2 >' 0$ such that

$$\lim_{n\to\infty}\frac{1}{\sigma_n-\rho_n}\sum_{k=\rho_n+1}^{\sigma_n}M_D\left(\frac{|\eta_k|_D}{\beta_1}\right) <' \infty$$

and

$$\lim_{n\to\infty}\frac{1}{\sigma_n-\rho_n}\sum_{k=\rho_n+1}^{\sigma_n}M_D\left(\frac{|\xi_k|_D}{\beta_2}\right) <' \infty.$$

Suppose,

$$\begin{aligned}
A &= \left\{\beta : \sum_{k=\rho_n+1}^{\sigma_n}M_D\left(\frac{|\eta_k+\xi_k|_D}{\beta}\right) \leq' 1\right\},\\
A_1 &= \left\{\beta_1 : \sum_{k=\rho_n+1}^{\sigma_n}M_D\left(\frac{|\eta_k|_D}{\beta_1}\right) \leq' 1\right\}\\
\textit{and}\quad A_2 &= \left\{\beta_2 : \sum_{k=\rho_n+1}^{\sigma_n}M_D\left(\frac{|\xi_k|_D}{\beta_2}\right) \leq' 1\right\}.
\end{aligned}$$

Let $\beta = (\beta_1 + \beta_2) \in A$, $\beta_1 = x_1' e_1 + x_2' e_2$, $\beta_2 = x_1'' e_1 + x_2'' e_2$ and $\beta = x_1 e_1 + x_2 e_2$.
Then

$$\begin{aligned}
& g(\eta + \xi) \\
& = \inf\left\{\beta : \sum_{k=\rho_n+1}^{\sigma_n} \left(M_D\left(\frac{|\eta_k + \xi_k|_D}{\beta}\right)\right) \leq' 1\right\} \\
& = \inf\{x_1 : \beta \in A\} e_1 + \inf\{x_2 : \beta \in A\} e_2 \\
& = \inf\{x_1' : \beta_1 \in A_1\} e_1 + \inf\{x_1'' : \beta_2 \in A_2\} e_1 + \inf\{x_2' : \beta_1 \in A_1\} e_2 + \inf\{x_2'' : \beta_2 \in A_2\} e_2 \\
& = \inf\{x_1' : \beta_1 \in A_1\} e_1 + \inf\{x_2' : \beta_1 \in A_1\} e_2 + \inf\{x_1'' : \beta_2 \in A_2\} e_1 + \inf\{x_2'' : \beta_2 \in A_2\} e_2 \\
& = \inf\left\{\beta_1 : \sum_{k=\rho_n+1}^{\sigma_n} M_D\left(\frac{|\eta_k + \xi_k|_D}{\beta_1}\right) \leq' 1\right\} + \inf\left\{\beta_2 : \sum_{k=\rho_n+1}^{\sigma_n} M_D\left(\frac{|\eta_k + \xi_k|_D}{\beta_2}\right) \leq' 1\right\} \\
& = g(\eta) + g(\xi).
\end{aligned}$$

Next we show continuity of the scalar multiplication.
Let $\gamma \in \mathbb{C}_2$ be a scalar.
Let

$$\begin{aligned}
B &= \left\{\beta : \sum_{k=\rho_n+1}^{\sigma_n} \left[M_D\left(\frac{|\gamma\eta_k|_D}{\beta}\right)\right] \leq' 1\right\}, \\
B_1 &= \{\beta_1 : \beta_1 e_1 + \beta_2 e_2 \in B\} \\
\textit{and} \quad B_2 &= \{\beta_2 : \beta_1 e_1 + \beta_2 e_2 \in B\},
\end{aligned}$$

where $\beta = \beta_1 e_1 + \beta_2 e_2$.
Then

$$\begin{aligned}
& g(\gamma\eta) = \inf B = e_1 \inf B_1 + e_2 \inf B_2 \\
& = \inf\left\{\beta_1 : \sum_{k=\rho_n+1}^{\sigma_n} \left[M_D\left(\frac{|\gamma\eta_k|_D}{\beta_1}\right)\right] \leq' 1\right\} e_1 + \inf\left\{\beta_2 : \sum_{k=\rho_n+1}^{\sigma_n} \left[M_D\left(\frac{|\gamma\eta_k|_D}{\beta_2}\right)\right] \leq' 1\right\} e_2 \\
& = \inf\left\{|\gamma|_D\tau_1 : \sum_{k=\rho_n+1}^{\sigma_n} \left[M_D\left(\frac{|\eta_k|_D}{\tau_1}\right)\right] \leq' 1\right\} e_1 + \inf\left\{|\gamma|_D\tau_2 : \sum_{k=\rho_n+1}^{\sigma_n} \left[M_D\left(\frac{|\eta_k|_D}{\tau_2}\right)\right] \leq' 1\right\} e_2 \\
& = |\gamma|_D \left[\inf\left\{\tau_1 : \sum_{k=\rho_n+1}^{\sigma_n} \left[M_D\left(\frac{|\eta_k|_D}{\tau_1}\right)\right] \leq' 1\right\} e_1 + \inf\left\{\tau_2 : \sum_{k=\rho_n+1}^{\sigma_n} \left[M_D\left(\frac{|\eta_k|_D}{\tau_2}\right)\right] \leq' 1\right\} e_2\right] \\
& = |\gamma|_D g(\eta),
\end{aligned}$$

where $\tau_i = \frac{\beta_i}{|\gamma|_D}, \quad i = 1, 2.$
Hence the theorem. □

4 Conclusion

In this work, we introduced different types of strong deferred Cesàro sequence spaces of bicomplex numbers using D-Orlicz function with respect to the D-norm. We also studied the algebraic, geometric and topological properties as well as some inclusion relations of these spaces. In the future, this work may help to construct different sequence spaces of bicomplex numbers related to D-Orlicz function and the D-norm.

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