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Vibration Analysis and Control: A Cantilever Beam with a Tuned Mass Absorber Under Harmonic Loading and Mass at the Free End

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Abstract — Vibration of a cantilever beam subjected to harmonic loading with a tip mass is mitigated by employing a tuned mass absorber at the free end. The beam is modeled using the Euler-Bernoulli theory to establish equations of motion for both the beam and the absorber. Stiffness and mass values ensuring zero deflection at the free end are determined via Galerkin's Method with a polynomial trial function. The analytical results are validated through FEM simulations in ANSYS Mechanical. The close agreement between computational and analytical outcomes establishes a foundation for advanced computational investigations.

Keywords — Tuned Mass Absorber, Harmonic Analysis, Galerkin's Method, FEM, Polynomial Shape Function

I. Introduction

Vibrations are a persistent challenge in mechanical and structural systems, often leading to fatigue, noise, instability, and even catastrophic failures. The cantilever beam, fixed at one end and free at the other, serves as a fundamental model for investigating such behaviours [1]. According to Euler–Bernoulli beam theory, its vibration response depends strongly on material and geometric properties [2]. When subjected to harmonic loading, these structures experience resonant oscillations [3], which are further amplified by additional concentrated masses at the free end-a realistic scenario in engineering practice. Such responses demand effective control strategies, and Tuned Mass Absorbers (TMAs) have emerged as passive devices capable of reducing resonant amplitudes [4,5]. The need to ensure structural safety, integrity, and functionality makes vibration control both highly relevant and urgent.

Prior research has extensively explored the dynamics of cantilever beams, establishing natural frequency predictions, mode shapes, and resonance behavior through analytical and computational methods. Ding et al. [5] demonstrated vibration isolation principles for elastic beams, while later studies extended the analysis to viscoelastic systems using harmonic balance methods [3]. Beléndez et al. [7] highlighted the influence of geometric nonlinearity, and Elkaranshawy et al. [6] advanced finite element formulations using Hamilton's principle. Computational work by Lian and Wang et al. [8,9] further improved nonlinear vibration solutions through Galerkin techniques. In parallel, vibration absorbers, dating back to Frahm's 1909 invention [10], have been successfully applied to skyscrapers [11], automobiles [12], and aerospace structures [13]. This body of research demonstrates both the importance and versatility of TMAs across disciplines.

Despite these advances, significant deficiencies remain. Most existing studies on vibration absorbers focus on discrete spring–mass systems [3,5], while comprehensive treatment of continuous systems like beams is limited. Moreover, relatively few investigations account for the combined effects of harmonic excitation and free-end concentrated masses, which alter resonance behavior in ways not fully captured by simplified models. The lack of systematic studies addressing these combined effects represents a critical gap in knowledge, leaving engineering practice without robust design guidelines for absorber implementation in such cases.

This research is highly relevant to specialists in structural dynamics, mechanical engineering, and vibration control. By addressing the coupled effects of harmonic excitation, free-end mass, and tuned mass absorber placement, the study provides new insights into dynamic response modification of flexible structures.

The purpose of this study is to analyze and control the vibration response of a cantilever beam with a concentrated free-end mass under harmonic excitation using a tuned mass absorber. Specifically, the research aims to develop analytical models based on Hamilton's principle, validate results through ANSYS simulations, and assess absorber performance under realistic loading conditions. By doing so,

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the study intends to close the current knowledge gap, extend the application of TMAs to continuous systems, and contribute practical solutions for structural vibration mitigation.

II. Mathematical Modelling

A. *without tuned mass absorber*

The beam of length L is fixed at one end and free at the other. At the free end, the mass M_1 is subjected to a harmonic force that varies sinusoidally with time, given by F_0 sin (ωt), where F_0 is the amplitude of the force and ω is the angular frequency. Due to this time-varying force, the beam experiences transverse (vertical) vibrations. The vertical displacement of the beam at any point x along its length and at any time t is denoted by w (x, t).

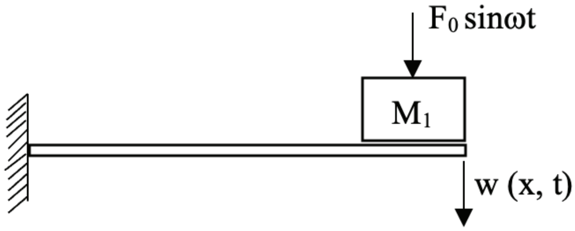

Fig.1 System without vibration absorber

The equation of motion of tip using Euler Bernoulli Beam Theory is:

$$\rho A\ddot{w} + M_1\ddot{w}\delta w\delta_d(x-L) + EIw^{iv} - F_0 sin\omega t\delta_d(x-L) = 0$$

Using assumed mode method using polynomial single mode approximation [1,2]:

Let w = $B\psi_1(x)\sin\omega t$ where $\psi_1(x) = x^4 - 4Lx^3 + 6L^2x^2$

Now,

$$\int_0^L \psi_1 \, R \, dx = 0$$

$$-\rho AB\omega^2 \int_0^L \psi_1(x)^2 dx - M_1 B\,\omega^2 \int_0^L \psi_1(x)^2\, \delta_d(x-L)\, dx + EIB \int_0^L \psi_1(x)\,\psi_1(x)^{iv}\, dx - F_0 \int_0^L \psi_1(x)\,\delta_d(x-L)\, dx$$

Also,

$$\int_0^L \psi_1(x)^2\, dx = \frac{104}{45}L^9$$

$$\text{'}\ \psi_1(x)^2|_{x=L} = (3L^4)^2 = 9L^8$$

$$\int_0^L \psi_1(x)\,\psi_1(x)^{iv}\, dx = \frac{144}{5}L^5$$

$$,\ \psi_1(x)|_{x=L} = 3L^4$$

$$B\left(-\frac{104}{45}\rho A\,\omega^2 L^9 - 9ML^8\omega^2 + \frac{144}{5}EI\,L^5\right) = 3F_0L^4$$

$$B = \frac{3F_0L^4}{-\frac{104}{45}\rho A\,\omega^2 L^9 - 9ML^8\omega^2 + \frac{144}{5}EI\,L^5}$$

$$\therefore w(x,t) = \frac{3F_0L^4\left(x^4 - 4Lx^3 + 6L^2x^2\right)}{-\frac{104}{45}\rho A\,\omega^2 L^9 - 9ML^8\omega^2 + \frac{144}{5}EI\,L^5}\sin\omega t$$

B. *WITH TUNED MASS ABSORBER*

The figure shows a cantilever beam with a tip mass M_1 subjected to a harmonic force. A spring-mass system, consisting of stiffness k and mass M_2, is attached at the free end to act as a dynamic vibration absorber system. Displacements w (x, t) and y (x, t) represent beam and absorber motion.

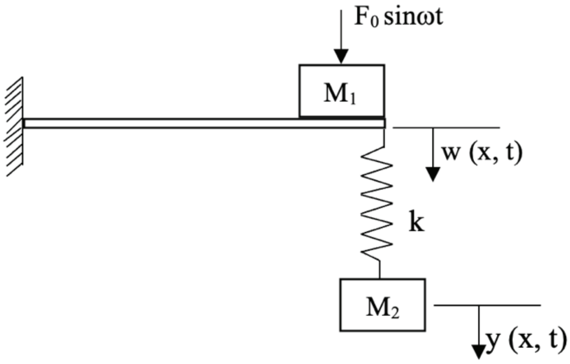

Fig.2 System without vibration absorber

The equation of motion is:

For primary system:

$$\rho A\ddot{w} + M_1\ddot{w}\delta_d(x-L) + EIw^{iv} - ky + kw\delta_d(x-L) - F_0 sin\omega t\delta_d(x-L) = 0 \quad (2)$$

For absorber system:

$$M_2\,\ddot{y} + ky - k\,\omega\,\delta_d(x-L) = 0$$

Now,

$$\int_0^L \psi_1\, R_1\, dx = 0$$

Let $w = B\psi_1(x)\sin\omega t$ where $\psi_1(x) = x^4\text{-}4Lx^3 + 6L^2x^2$

And $y = Y\,\psi_1\,(x)\sin\,\omega t$

$$-\rho AB\omega^2\int_0^L \psi_1(x)^2 dx - M_1 B\,\omega^2\int_0^L \psi_1(x)^2\,\delta_d(x-L)\,dx + EIB\int_0^L \psi_1(x)\,\psi_1(x)^{iv}\,dx -$$
$$F_0\int_0^L \psi_1(x)\,\delta_d(x-L)\,dx - k\,Y\int_0^L \psi_1(x)\,dx + kB\,\psi_1^2|_{x=L} = 0$$

$$B\left(-\frac{104}{45}\rho A\,\omega^2 L^9 - 9ML^8\omega^2 + \frac{144}{5}EI\,L^5\right) - \frac{6}{5}k\,L^5 Y = 3F_0L^4$$

For secondary mass:

$$\int_0^L \psi_1\,R_2\,dx = 0$$

$$\int_0^L -M_2\,\omega^2\,Y\,\psi_1(x)\,dx + kY\int_0^L \psi_1(x)\,dx - kB\int_0^L \psi_1(x)^2\,\delta_d(x-L)dx = 0$$

$$\left(-\frac{6}{5}M_2\,\omega^2\,L^5 + \frac{6}{5}kL^5\right)Y - (9L^8\,k)\,B = 0$$

Solving we get,

$$B = \frac{-270\,L^4\,F_0(k - M_2\,\omega^2)}{810\,\omega^2 L^8(-M_2 M_1\omega^2 + kM_1 + kM_2) - 720\,\omega^2\,EIL^6M_2 + 2592\,\omega^2\,EIL^5M_2 - 810\,k^2L^8 + 720\,EIKL^6 + 2025k^2L^7 - 2592\,EIK\,L^5 - 208M_2\omega^2\rho + 208\,kA\,\omega^2\rho}$$

$$Y = \frac{-2025\,L^7 kF_0}{810\,\omega^2 L^8(-M_2 M_1\omega^2 + kM_1 + kM_2) - 720\,\omega^2\,EIL^6M_2 + 2592\,\omega^2\,EIL^5M_2 - 810\,k^2L^8 + 720\,EIKL^6 + 2025k^2L^7 - 2592\,EIK\,L^5 - 208M_2\omega^2\rho + 208\,kA\,\omega^2\rho}$$

Finite Element Modelling (FEM) is carried out to numerically investigate the dynamic response of the cantilever beam subjected to harmonic loading, both in its original configuration and with the addition of a tuned mass absorber (TMA) at the free end. The primary purpose of this step is to verify the accuracy of the analytical solution derived using Hamilton's principle and to gain a detailed understanding of the displacement profile, phase response, and the overall effect of the absorber on vibration mitigation. FEM provides an effective means of visualizing how the absorber interacts with the beam, thereby validating the theoretical predictions and offering practical insights for design optimization.

A. *Model Setup*

The finite element model is created in ANSYS Mechanical to replicate the exact physical system considered in the analytical formulation. The beam is modeled as a three-dimensional solid with a rectangular cross-section having dimensions L = 0.5 m, b = 0.02 m, and h = 0.01 m. These dimensions are chosen to maintain consistency with the analytical model and ensure a representative dynamic response. The material properties are assigned as Young's modulus (E = 68.9 GPa), density (ρ = 2800 kg/m³), and Poisson's ratio (ν = 0.33), corresponding to an aluminum alloy beam.

Fig.3 Model Setup

The tuned mass absorber is implemented at the free end of the beam as a spring–mass system, where the spring has stiffness k and the absorber mass is M□. This system is connected directly to the beam tip to allow dynamic interaction between the absorber and the beam. The absorber parameters are chosen based on the analytical tuning conditions so that the absorber frequency coincides with the beam's excitation frequency, thereby achieving maximum vibration suppression.

B. *Meshing*

The beam is discretized using a structured mesh to transform the continuous beam into a finite number of elements that approximate its deformation behavior. A mesh size is selected carefully to strike a balance between computational efficiency and solution accuracy. A finer mesh is applied near the free end, where stress and displacement gradients are expected to be the highest, while a relatively coarser mesh is used near the fixed end where variations are minimal. This approach ensures that the computed displacements are sufficiently accurate while avoiding unnecessary computational cost.

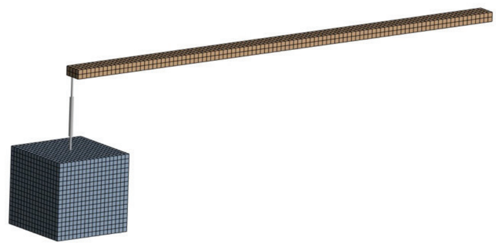

Fig.4 Meshing

C. *Boundary Conditions and Loading*

Appropriate boundary conditions are essential to correctly simulate the cantilever configuration. The clamped end (x = 0) is assigned a fixed support condition, restricting all translational and rotational degrees of freedom. This ensures zero displacement and zero slope at the support, consistent with the theoretical assumptions of the cantilever beam model.

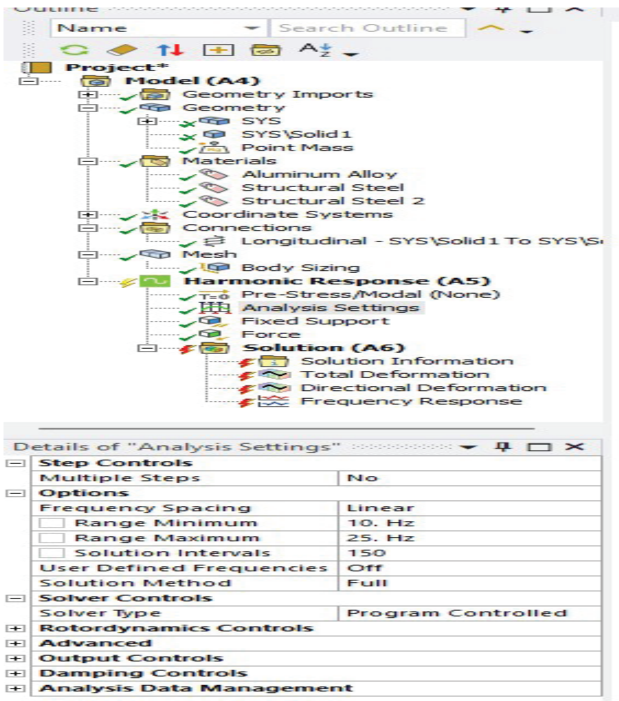

Fig.5 Boundary Conditions and Loading

D. *Harmonic Response Analysis*

After defining the geometry, material properties, boundary conditions, and loading, a harmonic response analysis is performed. The excitation frequency is swept over a range around the excitation frequency of interest to capture the steady-state response of the system. Two configurations are analysed:

- Without absorber: The baseline response of the bare beam is obtained, serving as a reference for evaluating the effectiveness of the absorber.
  With absorber: The tuned mass absorber is activated, and the resulting beam response is recorded.

From this analysis, the frequency response function (FRF), tip displacement, and phase relationship between the applied force and resulting displacement are extracted. These parameters provide valuable insight into how effectively the absorber reduces resonance-induced deflection and alters the phase of vibration.

III. RESULTS AND DISCUSSION

A. *Using Mathematical Model*

The mathematical model developed for the beam–absorber system provides a clear understanding of how the structure responds under harmonic excitation. By applying Hamilton's principle and formulating the governing equation of motion, the coupled dynamics between the primary beam and the vibration absorber are captured accurately. The deflection of the beam without the vibration absorber was found to be $1.2864*10^{-3}$m and the mass of absorber was found to be 3.177kg to reduce the vibration to zero.

B. *FEM Results*

- Without Absorber

For the given beam parameters and forcing conditions, the transverse displacement at the free end of the beam was found to be 1.2685mm. This indicates a significant vibration amplitude at the excitation frequency, suggesting the necessity of a vibration control mechanism.

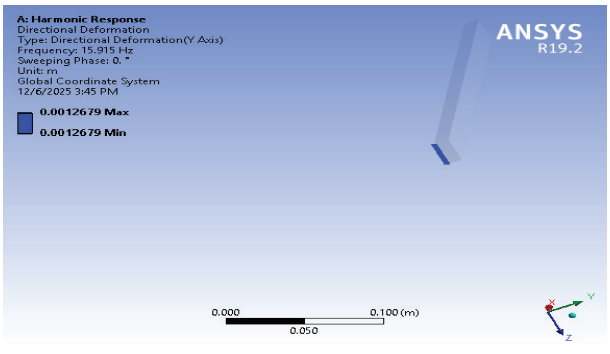

Fig.6 Deflection of free end of beam without vibration absorber

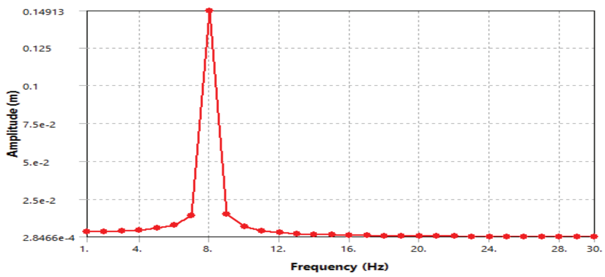

Fig.7 Frequency Response Analysis Without Absorber

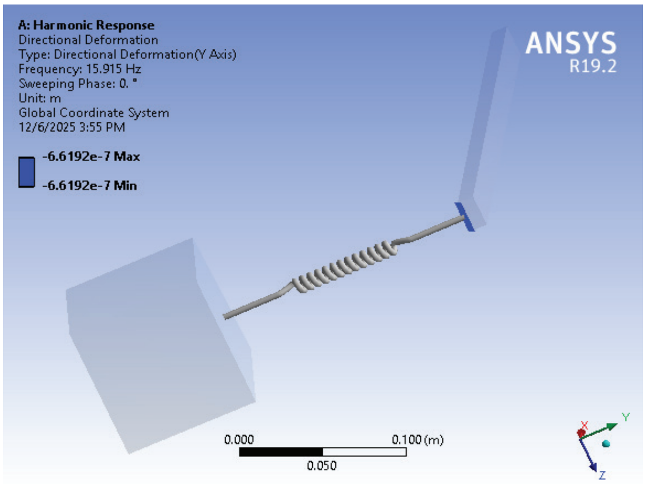

Fig.8 Deflection at beam free end with spring mass system

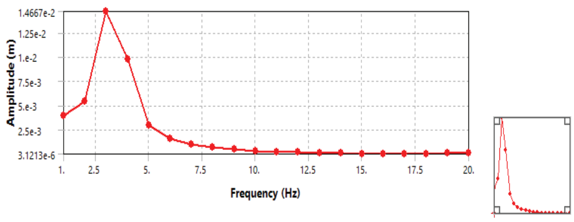

Fig.9 Frequency Response Analysis of System with Vibration Absorber

Introducing the vibration absorber as a mass - spring system coupled to the beam altered the dynamic response. For the tip of the beam, the displacement was calculated to be only 0.0010777mm, which is a significant decrease in comparison with no mass spring system.

As we can see from Fig X: Deflection of mass body, there is almost 0 deformation at the tip mass and maximum deflection at the introduced-mass body. The introduction of a mass-spring system helped to mitigate the vibration amplitude at the beam and the decrease in vibration amplitude confirms the absorber's effectiveness in reducing the beam's motion. Effect of Absorber Mass on Beam: Deflection For effective vibration suppression, the absorber's mass M_2 was optimized based on the given spring stiffness $K = 45933.33$ N/m. The analytical expression for M_2 in terms of K was derived, yielding an optimal mass of M□ = 3.177 kg for maximum vibration reduction.

The results confirm that the vibration absorber significantly reduces the beam's response, particularly when the absorber's natural frequency is tuned to the excitation frequency. The observed reduction in deflection highlights the importance of selecting appropriate absorber parameters, such as mass and stiffness, to optimize performance. A parametric study can further refine the design by investigating the influence of damping and absorber placement.

The analytical results can be further validated using FEM simulations in software like ANSYS or MATLAB, ensuring agreement between theoretical and numerical predictions. The developed model provides a practical foundation for designing passive vibration control systems in structural applications.

IV. Conclusion

The comparison between the analytical and simulation results for the cantilever beam without the vibration absorber shows excellent agreement, thereby confirming the accuracy of the developed mathematical model. The analytical free-end deflection of the beam is 1.2864×10^{-3} m, which closely matches the simulation result of 1.2685×10^{-3} m, indicating only a negligible difference between the two approaches. With the integration of the vibration absorber, the free-end deflection of the cantilever beam was reduced to 0.0010777mm in both the analytical and simulation results, confirming complete vibration suppression at the targeted location. This level of performance was achieved using an absorber mass of 3.177 kg and a spring stiffness of 45,933 N/m, as determined through the proposed analytical approach. The complete agreement between the analytical calculations and simulation results verifies the validity and efficiency of the designed vibration absorber in eliminating the harmonic response of the beam. Furthermore, this agreement confirms the reliability of the assumptions adopted in the vibration absorber analysis

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