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Understanding Tangent Plane and Normal Line using 3D Visualization

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Abstract

This designed based, Technological Pedagogical Content Knowledge grounded qualitative research work explores how 3D visualizations support students' conceptual understanding in mathematics. Involving a sample of 13 university students during 2024 cohort while learning a course "Differential Geometry", this study used observation checklist and classroom interactions notes as research tools. After analysis of collected data, the results showed that dynamic visualizations played a significant role in "formalized cognitive schema" as mental image, "improved conceptual understanding", and "directed students' attention and motivation toward dynamic media" effectively. So, this study established a connection between students' learning outcomes and their learning engagement through dynamic 3D visual simulations while learning tangent plane and normal.

Keywords: Javascript, DBR, Dynamic media Mathematica, Normal line, Tangent plane

Introduction

In the context of three-dimensional differential geometry, a surface refers to a two-dimensional manifold situated within the three-dimensional Euclidean space R^3. Locally, each point on the surface resembles a flat two-dimensional plane. More rigorously, surfaces can be represented in two distinct mathematical forms, for example (a) Explicit form and (b) Implicit form (Dhakal & Koirala, 2024; Pundir et al., 2021).

A surface is defined as locus of points (x,y,z) whose Cartesian coordinates x,y,z are function of two independent parameter, say u and v.

It is written as

"$x= f(u,v),\ y= g(u,v),\ z= h(u,v)$" (1)

A surface is defined as "locus of point (x,y,z) whose position with respect to origin O is function of two independent parameter u and v" (Dhakal & Koirala, 2024; Pundir et al., 2021).

It is written as

$\vec{r} = \vec{r}(u,v)$ *(2)*

A surface is defined as "locus of points *(x,y,z)* whose cartesian coordinate satisfy an equation" of the form

$F(x,y,z)=0$ *(3)*

Given $\vec{r} = \vec{r}(u,v)$ be a surface (Dhakal & Koirala, 2024; Pundir et al., 2021). The locus of points

$u = u(t)$ *(4)*

where v is constant is called a "u-parameter curve, u-curve, or v = constant curve". Similarly, the locus of points

$v = v(t)$ *(5)*

where u is constant is called a "v- parameter curve, v-curve, or u = constant curve".

Given $\vec{r} = \vec{r}(u,v)$ be a surface and P a point on it. Then it is mentioned in text that "there are infinitely many tangents at P" (Dhakal & Koirala, 2024; Pundir et al., 2021). Among them, the vectors

$\vec{r}_u = \frac{\partial \vec{r}}{\partial u}$ and $\vec{r}_v = \frac{\partial \vec{r}}{\partial v}$ (6)

are tangent to the u-curve and v-curve respectively. The tangent plane at P is the plane spanned by $\vec{r}_u$ and $\vec{r}_v$. The normal line at P is the line perpendicular to this tangent plane.

Problem Statement

In my class room practice, I observed that students often struggle to form a conceptual image of the tangent plane and normal line on a surface. This difficulty arises because, in their earlier study of space curves, students are introduced to the concepts of a tangent line and a normal plane, which are geometrically different.

In three-Dimensional Differential Geometry, for a given curve $\vec{r} = \vec{r}(t)$, the tangent is a line, and normals are also line, which is perpendicular to the tangent line. However, for a surface $\vec{r} = \vec{r}(u,v)$, tangent is a plane, and the normal is a line perpendicular to the tangent plane.

This "tangent normal dynamics" between curve and surface in three-Dimensional Differential Geometry and its understanding created a cognitive dissonance for students learning local properties of surfaces, especially when calculating fundamental directions; line tangent and normals. Based on classroom experience, it is observed that students very often attempted to apply their prior understanding of "unique tangent and infinite normals, which is on the curve" directly to surfaces, leading to misconception that surface has "unique normal and infinite tangents". Therefore, 3D visualization is is used to minimiza the seen misunderstandings.

Research Question

This research study helps to minimize congnitive conflicts that students encounter while learning geometric constructs of tangent plane and normal lines in higher mathematics. So the research question of this study is *"How do a design-based instructional intervention improve students' conceptual understanding of tangent plane and normal lines?"*

Research Objective

The research study of this *design-based experimental intervention is to understand*

"To explore students' conceptual understanding of tangent plane and normal lines while using design-based instructional intervention of 3D visualizations?"

Theoretical Framework

This study is grounded in the Technological Pedagogical Content Knowledge (TPACK) framework (Herring et al., 2014; Mishra et al., 2007), as theoretical framework. This framework provides a comprehensive lens to understand the value of technology integration in teaching pedagogy. In this research, technology integration is even important to edify complex mathematical concepts. TPACK is a name for the interplay among "content knowledge, which is known as CK", "pedagogical knowledge, which is known as PK", and "technological knowledge, which is known as TK" and together TPACK, valuing that effective pedagogy requires a pedagogically thoughtfull integration of these three components CK, PK, and TK (Anderson & Shattuck, 2012; Wang & Hannafin, 2005). Therefore, in this research, TPACK is considered as a guiding framework to address students' misconceptions in mathematics learning, and support student learning through meaningful conceptual visualizations, a thoughtful integration of content (CK), pedagogy (PK), and technology (TK) together.

Methods

The study utilized qualitative research design (Creswell et al., 2018; Denzin & Lincoln, 2018), particularly Design-Based Research (DBR) methodology (Anderson & Shattuck, 2012; Wang & Hannafin, 2005). This DBR is considered a flexible research framework, which can be adapted as needed during research work. Using this DBR approach, this study aims to design, implement, and refine instructional interventions that support students conceptual understanding in formaulating cognitive mental models, which is “schema” of "tangent plane and normal lines".

This study is carried out at the central department of education, Tribhuvan University Nepal among thirteen students during 2024 academic cohort. Data was collected using observation checklist and classroom interactions notes.

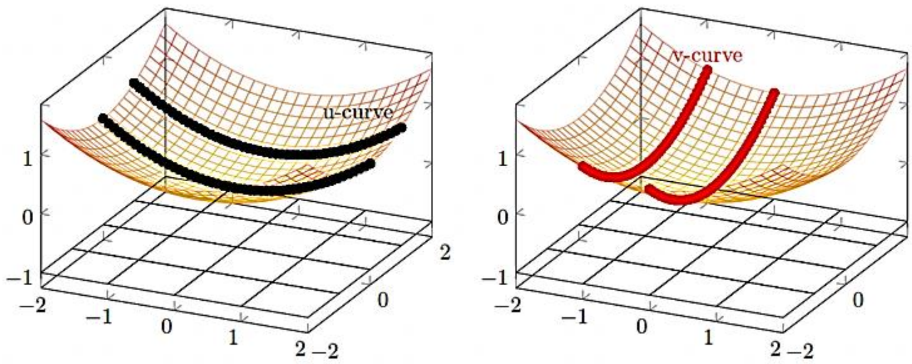
The phase used in this DBR are (a) analysis of practical problems that students are facing difficulties to have understanding with tangent plane and normal lines to the surface, (b) developing interactive visualizations of tangent plane and normal lines using first Mathematica and then developing the model in webpage using JavaScript code, (c) implementing video materials along with 3D visualizations through web page https://www.bedprasaddhakal.com.np/2024/07/tangent-plane-and-normal-line.html and (d) analyzing data using thematic coding of collected informations through throughchecklist and classroom interactions and reflections.

Results and Discussions

This qualitative design-based study investigates the potential application of 3D visualizations in teaching/learning of higher mathematics. In this particulalry study, Mathematica and JavaScript based interactive simulation and videos are used as research intervention. Then this research study explored how these 3D visualizations and instructor made videos can foster students learning understanding. After the analysis of collected data, this study given a research-based evidence that “dynamic media, which and together use of interactive simulation and video” can significantly support 21st century pedagogy by increasing student's engagement and interaction, both in physical and digital learning environment. Therefore, use of dynamic media are instrumental in clarifying mathematical concepts.

While designed and implemented the dynamic media of parametric curves, as shown in Figure 1, it illustrated the visualization of parametric curves on a surface defined by

$\vec{r} = \vec{r}(u,v) = (u,\ v,\ 1 + 0.5u^2 + 0.5v^2)$ (7)



(a) v =constant curve or u-curve (b) u=constant curve or v-curve

Figure 1: Parametric curves

In the Figure 1, the surface represents full parametric curves, while the black and red colored curves highlighted the *u*-parameter and *v*-parameter curves respectively.

The visualization given in Figure 1 allowed students to interactively "*manipulate the parameters and observe how the curves behave on the surface*". As a result, students were able to visualize the meaning of u = constant and v = constant curves, which became foundational knowledge in students understanding of tangent plane and normal lines. This is justified when a student responded that

"when I moved the sliders and saw the curve changing, I finally understood what a v = constant curve looks like on a surface. Before this, I thought parametric curves were just equations, a symbolic representation. Now I see them as paths on a surface, both symbolically and visually".

Based on the quotation, it is seen that first Mathematica and then JavaScript code shown, and its visualization helped learner to see how tangential directions are formed.

```
r[u_, v_] = {u, v, 1 + 0.5 u^2 + 0.5 v^2};
Manipulate[Show[
ParametricPlot3D[r[u, v], {u, -2, 2}, {v, -2, 2}],
ParametricPlot3D[r[u, v], {u, -2, 2}]],
{v, -2, 2}]
```

```
s=create('slider', [[-6, -6], [2, -6], [-2, 1, 2]], { name: ' v'});
create('parametricsurface3d', [
(u, v) => u,(u, v) => v,(u, v) => 0.5*u*u+0.5*v*v,
 [-2, 2], [-2, 2]]);
create('curve3d', [
 (t) => t , (t) =>  s.Value(), (t) => 1+0.5*t*t+0.5*s.Value()*s.Value(), [-2, 2]]);
```

This visualization helped students to understand the concept definition, given $\vec{r} = \vec{r}(u,v)$ be a surface. The locus of points $u = u(t)$ *(8)*

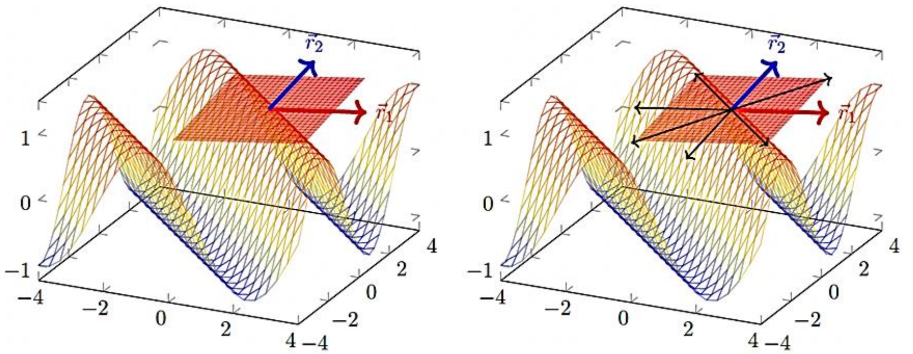
where v is constant is called a u-parameter curve, u-curve, or v = constant curve. Similarly, the locus of points, $v = v(t)$ *(9)*

where u is constant is called a v- parameter curve, v-curve, or u = constant curve.

Based on the understanding of tangential directions of parametric curves, tangent plane is defined ad it is formed by the directions of both type of parametric curves, locally, at each point. So, the students' response indicated a first milestone to understand "tangent normal dynamics". It is seen that visual and interactive tools significantly enhanced students' ability to form a "mental models of parametric curves" as prior schema of "tangent normal dynamics" of surface.

Next, given a surface $\vec{r} = \vec{r}(u,v)$ and P be a point on it, there are infinitely many tangents at P. Among such infinite tangents, $\vec{r}_1$ and $\vec{r}_2$ are the tangent to u-curve and v-curve respectively. Now, a plane spanned by $\vec{r}_1$ and $\vec{r}_2$ is called tangent plane at P. This concept is visualized by Mathematica first code, and then JavaScript.

```
r[u_, v_] = {u, v, Sin[u + v]};
r1[u_, v_] = Normalize[D[r[u, v], u]];
r2[u_, v_] = Normalize[D[r[u, v], v]];
n[u_, v_] = Cross[r1[u, v], r2[u, v]];
Manipulate[Show[
ParametricPlot3D[r[u, v], {u, -2, 2}, {v, -2, 2}],
ParametricPlot3D[{x, y, -5}, {x, -2, 2}, {y, -2, 2}],
ParametricPlot3D[r[p,q]+s*r1[p,q]+t*r2[p,q],{s,-1,1},{t,-1,1}],
Graphics3D[Point[r[p,q]]],
Graphics3D[Arrow[{r[p,q], r[p,q]+r1[p,q]}]],
Graphics3D[Arrow[{r[p,q], r[p,q]+r2[p,q]}]],
Graphics3D[Arrow[{r[p,q], r[p,q]+n[p,q]}]]
], {p, -2, 2}, {q, -2, 2}]
```



(a) Tangent plane by $\vec{r}_1$ and $\vec{r}_2$ (b) Infinite tangents to the surface

Figure 2: Tangent plane to the surface

As researcher, in this phase it is seen that, students were using JavaScript coded dynamic visualization of tangent plane and normal line. Because, one student responded that

"I used to think the tangent plane was just a flat surface touching the graph. Now I see it's made from two directions—the u-curves and v-curves themselves. Seeing the arrows (tangential directions) and the tangent plane together, it helped me to understand how the surface bends and how the tangent plane fits there".

From the quote, it is seen that interactive visualization improved students' conceptual understanding of tangent planes as geometric constructs derived from parametric curves, and a plane spanned by two tangents of parametric curves. It is the case where student manipulated the visualization as shown in Figure 2, it demonstrated the concept of the tangent plane to a surface at a point P, constructed from the tangent vectors $\vec{r}_1$ and $\vec{r}_2$, which are tangents to the u-curve and v-curve. Respectively. This visualized surface is defined parametrically by

$$\vec{r} = \vec{r}(u,v) = (uv, \sin(u+v)) \qquad (10)$$

In the Figure 2, it is seen that tangent plane is spanned by two vectors $\vec{r}_1$ and $\vec{r}_2$, and the normal vector is derived using the cross product of $\vec{r}_1$ and $\vec{r}_2$. Since Mathematics is propritary software, can not be deployed public webpage for user intrraction, therefore to enhance open use accessibility, the Mathematica code was implemented using JavaScript in HTML based web page. The JavaScript is open source, and it works in web platforms.

Finally, as a researcher, I have visualized the concept of the normal vector to a surface at a given point *P*, which is cross product of the tangent vectors $\vec{r}_1$ and $\vec{r}_2$. In visualization (a), the tangent plane is formed by the vectors $\vec{r}_1$ and $\vec{r}_2$, while in (b), the normal vector $\vec{N}$ is shown as perpendicular to both $\vec{r}_1$ and $\vec{r}_2$, and hence to the tangent plane. This geometric relationship was dynamically visualized using in JavaScript for interactive exploration via a webpage.

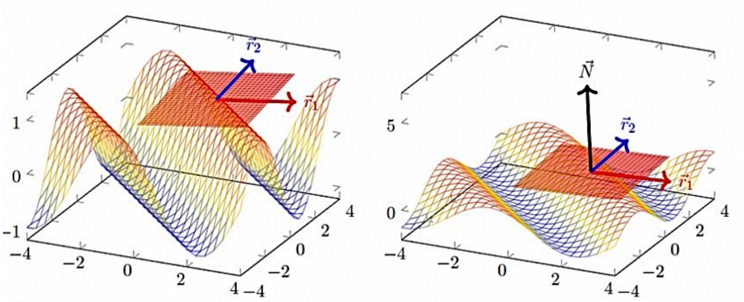
While student explored during the live class, one student responded that
"I didn't know the normal line was built from two tangential directions. Seeing the normal line, which it perpendicular to the tangent plane made, it is now clear for me. When I rotated the 3D graph and saw the normal line, it helped me to understand how it relates to the surface and the tangent plane".

The students' response as a quote evidenced that interactive visualizations enhanced their spatial reasoning and built a conceptual clarity through "cognitive mental model" regarding the tangent plane and normal line. In the textbook, the normal vector is defined as, given a surface $\vec{r} = \vec{r}(u,v)$ and P be a point on it. Then $\vec{r}_1 \times \vec{r}_2$, is a line perpendicular to the tangent plane, so, $\vec{r}_1$ and $\vec{r}_2$, is normal line at *P*. Since, $\vec{r}_1 \times \vec{r}_2$ is normal line, we denote the unit vector along normal line by $\vec{N}$, which is defined as

$$\vec{N} = \frac{\vec{r}_1 \times \vec{r}_2}{|\vec{r}_1 \times \vec{r}_2|} \tag{11}$$

or $$\vec{N} = \frac{\vec{r}_1 \times \vec{r}_2}{H} \text{ where } |\vec{r}_1 \times \vec{r}_2| = H \tag{12}$$

when student used the visualization, the concept definition as mentioned in number of books, for example, Koirala and Dhakal; Pundir et al (Dhakal & Koirala, 2024; Pundir et al., 2021), the "concept-definition of tangent plane and normnal line" is explained by the dynamic media; the visualization and video, as "concept-image".



(a) Tangent plane by $\vec{r}_1$ and $\vec{r}_2$ (b) Normal line along $\vec{r}_1 \times \vec{r}_2$

Figure 3: Normal line to the surface

As purposed in the theoretical framework, the TPACK grounded to help researcher in digital simulation of mathematical concept. It has been a robust framework because "technology helped in pedagogy for higher mathematics, which is the essence of TPACK use" (Herring et al., 2014; Mishra et al., 2007). In higher mathematics, like multivariable calculus and Differential Geometry, the "tangent normal dynamics" between curve and surface posed a significant cognitive challenges for learners. Now using the 3D visualization, and thoughtful integration of TPACK as suggested by Mishra et al. (2007), it is found that "tangent normal dynamics" are more intuitive for learners. However, the research also acknowledges the pedagogically thoughtful integration of TPACK, not just merely use of TPACK, otherwise the technology itself remains insufficient to visualize mathematical abstraction.

In this study, it is seen that, when learner used the dynamic visualizations, which forst the researcher computed in Mathematica then deployed in HTML based web page via JavaScript, students were able to simulate it, and formalize a conceptual understanding of the geometric meaning behind the equations of the tangent plane and normal line. From the response of students, it is conformed that student visualized the figure of webpage as concept-image, which helped them to understand the concept-definition of tangent plane, which is

$$[R - \vec{r}, \vec{r}_1 , \vec{r}_2] = 0 \tag{13}$$

and equation of normal line, which is

$$R = \vec{r} + \lambda(\vec{r}_1 \times \vec{r}_2) \tag{14}$$

The both equation, which is also the response of students, it represents that learners were able to formalize the concept "normal line is a line perpendicular to the tangent plane", which os constructed from the cross product of the two tangent vectors $\vec{r}_1 \times \vec{r}_2$. It is additionally justified by the response of a student, who mentioned that *"...before, I just use to memorize the formula, now I see how the plane is built from two directions and the normals, it is now easy for me to remember the formula"*.

From the data and results presented above, it can be argued that "3D visualization made the symbolic equation of "tangent normal dynamics" more student centered and beyong "just symbols like mathematics". The result showed that "tangent plane and the normal line" visualization together helped learners to understand "how tangent plane and normal line relate each other". In literature, it has been mentioned that "dynamic visualization and simulations play an important role in bridging symbolic representations and spatial understanding" (Erol & Saygı, 2024; Sulastri et al., 2021).

Based on this argument, it is justified that "3D visualization in this research, established a learning connection between concept image and concept definition of tangent plane and normal line".

These findings are also similar to some of the research works, for example Alkan & Ada, Bos & Wigmans, and Caniglia & Meadows, which emphasized that "effectiveness use of interactive technologies foster problem-solving abilities in abstract mathematical contexts, and it further helps in enhancing conceptual comprehension of mathematical ideas"(Alkan & Ada, 2024; Bos & Wigmans, 2025; Caniglia & Meadows, 2025). Therefore, the research work is worthy.

This study has made a research contribution, which is not just a technology, but a pedagogically thoughtful technology integration, it is also explained by TPACK framework. Therefore, along with the technology advancement, technology innovation, it is wise to do a systematic and pedagogically thoughtful DBR based application to peruse dynamic media to enhance students understandings. This research work also acknowledge the value that "DBR is iterative refinement of works based on user feedback and interaction", which is also discussed in number of research works, for example dynamic Multimedia Development Life Cycle (MDLC) methodology (Anderson & Shattuck, 2012; Wang & Hannafin, 2005). Hence, this study is resulted with a quote that "pedagogically thoughtful use of dynamic media is instrumental"

Finding and Conclusion

Based on the research results and discussions, it is found that 3D visualizations is supportive to enhance learners conceptual understanding in higher mathematics. Particularly in this study, it is found that "dynamics of tangent plane and normal line" to a surface is edified, when learner use 3D visualizations. The evidence is justified when students were able to use the visualization and "grasp the core concept definition vis concept image". Therefore, this finding highlights the value of TPACK so that learner became able to steped ahead from symbolic memorization to visual reasoning in the cognitive learning trajectories via concept definition to concept image.

The pedagogic integration of 3D visualization, and using this qualitative design-based research (DBR), this research work exemplified the "thoughtful pedagogical value of using innovative technologies" in the field of mathematics learning, which actually helped learner to sync conceptual and cognitive difficulties to "schema" as mental image or concept-image.

Therefore, this piece of work recommends for scholars and readers to incorporate interactive and immersive technologies into their own teaching and learning. However, further and similar research work could be done to explore the "bigger impact of 3D visualizations in mathematics education".

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