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The Evolution of Euler's Summability in Mathematical Arena

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Abstract

Euler was the first mathematician who studied infinite series in depth, especially focusing on divergent series. He asserted that divergent series must have some definite value. He became successful in this direction by finding a system to sum such series. In these days, Euler's summation methods are popular with some modifications in different names like Cesaro, Abel, Hölder, Nörlund, Hausdorff matrices, Borel summation, and so on. In this paper, I discuss some of Euler's work on summation methods of infinite series associated with the Euler-Maclaurin summation formula. We also examine Euler's effort on the solution to the Basel problem and his devotion to Grandi's series with some examples.

Keywords: Approximation**,** Bassel problem, Euler-Maclaurin formula, Infinite series, Summation methods

Introduction

The study of infinite series has been popular in mathematical analysis. The sum of an infinite series has been calculated by applying different mathematical theories. In the study of infinite series, we obtained that there are two popular forms known as convergent series and divergent series, depending on whether they have a sum or not. The study of infinite series took its emerging path since 2000 B.C. in search of the value of π. Based on the history of mathematics, it is observed that the Babylonians used the infinite series as the basic concept in many cases, like the counting number system and its elaboration. But there was no rigorous study, and it was propounded within the boundary of Geometry. Archimedes of Ancient Greece developed the concept of infinite series, especially geometric series. He also estimated the value of π around 250 BC in search of geometrical properties of different shapes. Beginning with the regular hexagons inscribed in a circle and doubling the number of sides repeatedly until he obtained a regular polygon of 96 sides, and he declared that $3\frac{10}{17} < \pi < 3\frac{1}{7}$ (Beckmann, 1971). It is the fruitful initiation of finding the value of π in association with the infinite series. The Indian mathematician Madhava of Sangamagrama (c. 1350 – c. 1425) and his school developed infinite power series for trigonometric functions like sine and cosine, centuries before their rediscovery in Europe. Madhava, for the first time, proposed the relation that $\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right)$ as an infinite series. Madhava also deduced the inverse tangent function as $tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}$+... (-1< $x \leq 1$) .It was later discovered in Europe by James Gregory (1671) and Gottfried Wilhelm Leibniz (1674). Hence, it was known as the Gregory-Leibniz series, but nowadays it is referred to as the Madhava series (Swain, 2021). In the 18th century, Leonhard Euler did much work on infinite series and their sums. In the 20th century, Godfrey Harold Hardy, John Edensor Littlewood, and Ramanujan brought different ideas in summing the infinite series. Especially, they raised different ways of summability methods treating with divergent series (Paudyal, 2013; 2014). Guido Grandi proposed an infinite series 1-1+1-1+1… in 1703 and raised an issue to find its sum. Many mathematicians proved it as a divergent series, and they claimed that it has no sum. Although mathematically divergent, such series were often assigned practical values by researchers, especially within physics, engineering, and related applied sciences. For this reason, Euler investigated divergent series, proposed alternative methods to assign them values, and thereby developed a new branch of mathematics known as summability theory. English mathematician G.H. Hardy and Indian mathematician Ramanujan also studied divergent series in this direction as part of mathematical analysis and number theory (Paudyal, 2014).

Euler's Summability Methods

A summation (or summability) method is a function that assigns values to infinite series. Euler attempted some special series. Euler treated infinite series different from the conventional method of finding the limit of a sequence of the n^{th} partial sum of the given series. Euler transformed the given infinite series into a different form and proved its convergence to a sum. This sum is called the Euler sum of the original series. These methods give some definite values to those series which are usually said to be divergent series in classical sense. These are known as Euler's summation methods. Euler summation method can also be used to speed the convergence of series. Such method is also known as Euler's transformation for the approximation of the sum of an infinite series.

Euler's approach to Grandi's Series:

Let us consider the infinite series $\sum_o^\infty a_n$ where $a_n = (-1)^n$. Thus, this series looks like

$$\sum_o^\infty (-1)^n = 1-1+1-1+1-1+\ldots \qquad (2.1)$$

which is known as Grandi's series.

The series (2.1) is an infinite series that diverges, and hence it has no sum in the classical sense. By using Euler's transformation, this series (2.1) is treated in the following way.

Here, the series (2.1) is an alternative series for which Euler treated using the approach

$S = \frac{A_1}{2} - \frac{\alpha}{4} + \frac{\beta}{8} - \frac{\gamma}{16} + \frac{\delta}{32} - \ldots$ where $A_1, \alpha, \beta, \gamma, \delta \ldots$ are the first difference, second difference and so on. In (2.1), $a_0 = 1$ and first difference $\Delta a_0 = 0$, second difference $\Delta^2 a_0 = 0$,third difference $\Delta^3 a_0 = 0$ and so on.

$\therefore S=1-1+1-1+1-1+\ldots = \frac{a_0}{2} - \frac{\Delta a_0}{4} + \frac{\Delta^2 a_0}{8} - \ldots = \frac{1}{2} - \frac{0}{4} + \frac{0}{8} - \ldots = \frac{1}{2}$ (Varadrajan,2007)

Euler systematized it by publishing a paper in 1775 that the sum of alternating series $\sum_{n=0}^{\infty}(-1)^n a_n$ is given by, $\sum_{n=0}^{\infty}(-1)^n a_n = \sum_{n=0}^{\infty}(-1)^n a_n . \frac{\Delta^n a_0}{2^{n+1}}$ (Kline,1983)

(2.2)

Euler also developed a more convincing relation for the approximation of the sum of divergent series. A series $\sum_{n=0}^{\infty} a_n$ is summable by means of the Euler method to the sum S if $\lim_{n\to\infty} \frac{1}{(q+1)^n} \sum_{k=o}^{n} \binom{n}{q} q^{n-k} S_k = S$ where $q > -1$ *and* $S_k = \sum_{n=0}^{k} a_n$. Now, let's apply this method to Grandi's series (2.1). Consider the series (2.1), the partial sums of which are 1, 0, 1, 0, ...It shows, by definition, that it is a divergent series and it has no sum. But Euler attempted it in the following way. Applying Euler's summation (E, q) method, we can have

Let $t_n = \frac{1}{(q+1)^n} \sum_{k=o}^{n} \binom{n}{k} q^{n-k} S_k$ (2.3)

$\Rightarrow t_1 = \frac{1}{(1+1)^1} \{C(1,0)1^1 S_0 + C(1,1)1^0 S_1\} = \frac{1}{2}$

$t_2 = \frac{1}{(1+1)^2} \{C(2,0)1^2 S_0 + C(2,1)1^1 S_1 + C(2,2)1^0 S_2\} = \frac{1}{2}$

$t_3 = \frac{1}{(1+1)^3} \{C(3,0)1^3 S_0 + C(3,1)1^2 S_1 + C(3,2)1^1 S_2 + C(3,3)1^0 S_3\} = \frac{1}{2}$

$t_4 = \frac{1}{(1+1)^4} \{C(4,0)1^4 S_0 + C(4,1)1^3 S_1 + C(4,2)1^2 S_2 + C(4,3)1^1 S_3 + C(4,4)1^0 S_4\} = \frac{1}{2}$

and so on for q = 1; $S_k = \begin{cases} 1 \ ; \ if\ k\ \ even \\ 0 \ ; \ if\ k\ is\ odd \end{cases}$

$\therefore \lim_{n\to\infty} t_n = \lim_{n\to\infty} \frac{1}{(q+1)^n} \sum_{k=o}^{n} \binom{n}{q} q^{n-k} S_k = \frac{1}{2}$

In this case, the transformed sequence t_n converges to 1/2. Therefore, the series 1 - 1 + 1 - 1 + ... is (E, 1) summable to $\frac{1}{2}$ .It means its (E,1) sum is $\frac{1}{2}$

In his 1746 paper "De seriebus divergentibus" (read in 1754 and published in 1760), Euler investigated the summation of divergent series including the Grandi's series. He considered the geometric series 1 - a + a^2 - a^3 + ... which converges to 1/(1+a) for $|a| < 1$. Extending this formula formally, Euler substituted a =1, and obtained $\frac{1}{2}$ which is the Euler sum of Grandi's series. Later, in his paper "On Divergent Series" published in 1754, Euler elaborated further on these ideas, presenting summability methods as his original contributions to the theory of series (Kline,1983; Varadrajan,2007).

Euler's Summation Formula

In 1734, Leonhard Euler developed a summation formula and soon used it to estimate the sum of the reciprocal of squares of all-natural numbers and declared that the sum is $\frac{\pi^2}{6}$. This problem popularly known as Basel problem was proposed by Pietro Mengoli in 1644. It was solved by Euler in 1734 was really an iconic work. Consequently, Euler elaborated the Summation formula in connection with Bernoulli numbers and applied it to various issues of mathematics and physics. He applied it to estimate the sum of finite terms of harmonic series, the gamma constant, sums of logarithms and Stirling's series etc. Both the mathematicians Leonhard Euler and Colin Maclaurin developed same formula independently in 1735. It became popular as Euler-Maclaurin Summation Formula. It connects integral value and the related sum. In general, it is said that Euler used Euler-Maclaurin Summation Formula to compute slowly converging infinite series while Maclaurin used it to calculate integrals. It is very tedious work to find the direct sum of many terms or infinite terms of a series but Euler did it easily by developing such summation formula to find the approximate sum. Then, Euler-Maclaurin summation formula is in the form,

$$\sum_{k=u}^{v} f(k) = \int_u^v f(x)dx + \frac{f(v)+f(u)}{2} + \sum_{k=1}^{p} \frac{B_{2k}}{(2k)!}[f^{2k-1}(v) - f^{2k-1}(u)] + R_p$$

(Ehenberg,1966; Paudyal and Mishra,2020; Hall,1964)

Here f(x) is a real valued continuous function defined between u and v while R_p *is* the error for suitable value of p (a natural number). They couldn't calculate error at that time. It was calculated later by French Mathematician S.D. Poisson (1781-1840). If we assume $I=\int_u^v f(x)dx$ and $S = \frac{f(v)+f(u)}{2} + f(u+1) + f(u+2) + \cdots + f(v-2) + f(v-1)$,then Euler-Maclaurin formula provides the expression S-I $= \sum_{k=1}^{p} \frac{B_{2k}}{(2k)!}[f^{2k-1}(v) - f^{2k-1}(u)] + R_p$, p is a suitable natural number and B_{2k} are Bernoulli numbers. It is also noted that Euler had already estimated the Bernoulli numbers named by $B_0 = 1,\ B_1 = 1/2, B_2 = \frac{1}{6}$, $B_4 = \frac{-1}{30}$, $B_6 = \frac{1}{42}$, $B_8 = \frac{-1}{30}$, $B_{10} = \frac{5}{66}$ and so on.

Example: Find the sum of series given by $\sum_{n=1}^{5} 3^n$ by using Euler- Maclaurin summation formula.

Sol^n: Here our task is to find $\sum_{n=1}^{5} 3^n$.

Consider a function $f(x) = 3^x$ which is continuous on [1,5]

Step I

Find, $\int_u^v f(x)dx = \int_1^5 3^x dx = \left[\frac{3^x}{ln3}\right]_1^5 = \frac{240}{ln3}$

Step II

Find, $\frac{f(u)+f(v)}{2} = \frac{3^5-3^1}{2} = 123$

Step III

Find derivatives, $f'(x) = 3^x\ ln3 \Rightarrow f'(5) - f'(1) = 3^5\ ln3 - 3^1\ ln3 = 240ln3$

$f''(x) = 3^x\ (ln3)^2 \Rightarrow f''(5) = 3^5\ (ln3)^2$ and $f''(1) = 3^1\ (ln3)^2$

$f'''(x) = 3^x\ (ln3)^3 \Rightarrow f'''(5) - f'''(1) = 3^5\ (ln3)^3 - 3^1\ (ln3)^3 = 240(ln3)^3$

$f^{iv}(x) = 3^x\ (ln3)^4 \Rightarrow f^{iv}(5) = 3^5\ (ln3)^4$ and $f^{iv}(1) = 3^1\ (ln3)^4$

$f^{v}(x) = 3^x\ (ln3)^5 \Rightarrow f^{v}(5) - f^{v}(1) = 3^5\ (ln3)^5 - 3^1\ (ln3)^5 = 240(ln3)^5$

Now we apply Euler-Maclaurin summation formula,

$$\sum_{k=u}^{v} f(k) = \int_u^v f(x)dx + \frac{f(v)+f(u)}{2} + \sum_{k=1}^{p/2} \frac{B_{2k}}{(2k)!}[f^{2k-1}(v) - f^{2k-1}(u)]$$

$$\sum_{k=1}^{5} 3^k = \int_1^5 3^x dx + 123 + \sum_{k=1}^{4} \frac{B_{2k}}{(2k)!}[f^{2k-1}(v) - f^{2k-1}(u)]$$

$$= \frac{240}{ln3} + 123 + \frac{B_2}{2!}[f'(5) - f'(1)] + \frac{B_4}{4!}[f'''(5) - f'''(1)] + \frac{B_6}{6!}[f^{v}(5) - f^{v}(1)]$$

$$= \frac{240}{ln3} + 123 + \frac{1/6}{2!} \times 240\ ln3 + \frac{-1/30}{4!} \times 240(ln3)^3 + \frac{1/42}{6!} \times 240(ln3)^5$$

=363.0003719 (Using Desmos Scientific Calculator)

It is noted that the actual sum of given series is 363. If we add some terms up to 2k-1 = 9, we get

$\sum_{k=1}^{5} 3^k = \frac{240}{ln3}$+123+$\frac{1/6}{2!} \times 240\ ln3$+$\frac{-1/30}{4!} \times 240(ln3)^3$+$\frac{1/42}{6!} \times 240(ln3)^5$+$\frac{-1/30}{8!} \times 240(ln3)^7$+$\frac{1/66}{10!} \times 240(ln3)^9$ =362.999991 (More accurate than above).Here ,we remain to calculate the error by using formula given by Poisson.

So, we say that Euler Maclaurin formula connects the summation and integral values to find the finite and infinite sum under the given conditions. Many mathematicians, later modified this formula in various forms for the systematic applications in other fields like Physics and Engineering where one needs new ideas of summation.

Euler's Approach of Solution to Basel problem

Let $f(x) = x^2 - 3x + 2$ be a polynomial function. It has two zeros such that x=2 and x=1. Then it is clear that this function can be expressed as the products of two factors as (x-2) and (x-1). It means we can write,

$$f(x) = (x-2)(x-1) \quad \Rightarrow f(x) = 2\left(1-\frac{x}{2}\right)(1-\frac{x}{1})$$

By assuming similar logic, Euler claimed that it holds true in sine function too. He used that sine function has infinitely many zeros $0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$ and hence one can express the function sinx as the infinite products of linear factors as,

$$sinx = k(x-0)(x-\pi)(x+\pi)(x-2\pi)(x+2\pi)(x-4\pi)(x+4\pi)\ldots$$

$$= kx(x^2-\pi^2)(x^2-4\pi^2)(x^2-16\pi^2)\ldots$$

$$\Rightarrow \frac{sinx}{x} = k(x^2-\pi^2)(x^2-4\pi^2)(x^2-16\pi^2)\ldots \quad (2.3.1)$$

$$\Rightarrow \lim_{x\to 0}\frac{sinx}{x} = k(-\pi^2)(-4\pi^2)(-16\pi^2)\ldots$$

$$\Rightarrow 1 = k(-\pi^2)(-4\pi^2)(-16\pi^2)\ldots$$

$$\therefore k = \frac{1}{(-\pi^2)(-4\pi^2)(-16\pi^2)\ldots}$$

Substituting the value of k in (2.3.1)

$$\frac{sinx}{x} = (1-\frac{x^2}{\pi^2})(1-\frac{x^2}{4\pi^2})(1-\frac{x^2}{9\pi^2})(1-\frac{x^2}{16\pi^2})(1-\frac{x^2}{25\pi^2})(1-\frac{x^2}{36\pi^2})(1-\frac{x^2}{49\pi^2})\ldots$$

$$\Rightarrow \frac{sinx}{x} = \prod_{n=1}^{\infty}(1-\frac{x^2}{n^2\pi^2}) \quad (2.3.2)$$

which is known as a product formula.

In other hand, Euler applied the Taylor's expansion of sine function given by

$$sinx = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots$$

$$\frac{sinx}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} - \ldots \quad (2.3.3)$$

From (2.3.2) and (2.3.3) it is obtained that

$$1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\frac{x^8}{9!}-\ldots\ =\prod_{n=1}^{\infty}(-\frac{x^2}{n^2\pi^2}) \qquad (2.3.4)$$

Expanding the right-hand side of (2.3.3) by finding the infinite products and equating the coefficients of x^2 on both sides we get

$$-\frac{1}{3!}=-\sum_1^{\infty}\frac{1}{n^2\pi^2}$$

$$\Rightarrow \frac{1}{\pi^2}\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\frac{1}{7^2}+\frac{1}{8^2}+\frac{1}{9^2}+\frac{1}{10^2}+\cdots\right)=\frac{1}{3!}$$

$$\Rightarrow\ 1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\frac{1}{7^2}+\frac{1}{8^2}+\frac{1}{9^2}+\frac{1}{10^2}+\cdots=\frac{\pi^2}{6} \qquad (2.3.5)$$

which is the required solution to the Basel problem. Euler attempted this problem in 1735 and published. This solution was also published in 1741 in more prescribed form and he is supposed to be a successful mathematician. Now we can also have from (2.3.5)

$$\left(\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\frac{1}{8^2}+\cdots\right)+(1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\frac{1}{11^2}+\cdots)=\frac{\pi^2}{6}$$

$$\Rightarrow\frac{1}{4}(\ \ 1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\frac{1}{7^2}+\cdots)+(1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\frac{1}{11^2}+\cdots)=\frac{\pi^2}{6}$$

$$\Rightarrow\frac{1}{4}\times\frac{\pi^2}{6}+(1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\frac{1}{11^2}+\cdots)=\frac{\pi^2}{6}$$

$$\Rightarrow\frac{\pi^2}{24}+(1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\frac{1}{11^2}+\cdots)=\frac{\pi^2}{6}$$

$$\Rightarrow\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\frac{1}{11^2}+\cdots\right)=\frac{\pi^2}{6}-\frac{\pi^2}{24}$$

$$\therefore 1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\frac{1}{11^2}+\cdots=\frac{\pi^2}{8} \qquad (2.3.6)$$

Also**,**

$$\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\frac{1}{8^2}+\cdots=\frac{\pi^2}{24} \qquad (2.3.7)$$

On the basis of Euler -Maclaurin summation formula and Euler's solution to Basel problem, Paudyal and Mishra (2020) proposed a generalization to find the sum of infinite series stating that "For any function f(x) which is a continuous, positive and decreasing and defined on [0, ∞) such that its infinite sum formed by this function can be approximated by the relation

$$S\cong\sum_1^k f(n)+\int_{\frac{2k+1}{2}}^{\infty}f(x)\,dx \qquad (2.3.8)$$

In 1966, D.O. Ehrenburg had already established that the integral $\int_{\frac{1}{2}}^{\infty}f(x)$ exists for $\frac{1}{2}\le x<\infty$ whenever f(x) is monotonic, continuous and has continuous derivatives. Consequently, he expressed the same relation (2.3.8) for the approximation of sum of infinite series in this particular case. Now we use this relation to solve the Basel problem $\sum_1^\infty \frac{1}{n^2}$ .Consider f(x) = $\frac{1}{x^2}$ which is positive, decreasing and continuous function defined on (0, ∞). So we can have

$$S \cong \sum_1^k f(n) + \int_{\frac{2k+1}{2}}^\infty f(x)\, dx$$

$$= \sum_1^{10} \frac{1}{n^2} + \int_{\frac{21}{2}}^\infty \frac{1}{x^2}\, dx$$

$$\cong 1.549767731 + 0.09523809524\text{Con}$$

$$= 1.64500582624$$

which is nearer to the actual value developed by Euler equal to $\frac{\pi^2}{6}$

We observed that this formula (2.3.8) is supposed to be one of the very simple and valuable tools to find the approximation sum of such infinite series provided the integral exists.

Conclusion

In our study, we found that Euler made significant contributions to the study of infinite series. He asserted that Grandi's series can be solved by finding its (E,1) sum, which equals $\frac{1}{2}$. Euler also solved the Basel problem and determined its value to be $\frac{\pi^2}{6}$. The very interesting aspect of his solution is that he applied Taylor's series to expand sinx and then expressed this function as an infinite product of linear factors by identifying the zeros of the sine function. By equating the corresponding coefficients of powers of x, he not only obtained the required solution to the Basel problem but also derived further important consequences. We also applied the Euler-Maclaurin summation formula to find the finite sum of a series of positive terms, which is also useful for evaluating integrals that cannot be solved by usual processes. Consequently, we observed a strong connection among infinite series, summability theory, and integrals. Through examples, we also showed that Euler's solution to the Basel problem stands as a pioneering and remarkably accurate work. Likewise, Euler employed this formula in the study of the zeta function, the gamma function, and related areas. Moreover, his work on infinite series laid a strong foundation for the development of summability theory. We conclude that further work in this direction could show more clearly how Euler's techniques became the groundwork for rigorous summability theory.

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