Correlation Between the Z-Transform and Laplace Transform in Discrete-Time Signals

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Abstract

This paper focuses on introducing Z-transform and Laplace transform, as well as exploring the relationship between them. The z-transform's function in the analysis of discrete-time single and LTI systems is reviewed, as is the role of the Laplace transform in the study of continuous-time signals and L.T.I., as well as what the particular region of convergence represents. The area of convergence has been illustrated graphically, and the relationship is examined.

This paper starts by deriving the z-transform from the Laplace transform applied to a discrete-time signal.

1. Introduction

The Z-transform is a crucial mathematical tool for system design, analysis, and monitoring, much like the Laplace transform. The z-transform, which is a generalization of the Fourier transform of a sampled signal, is the discrete-time counterpart of the Laplace transform. The z-transform, like the Laplace transform, provides information about the stability, steady state behavior, and transient behavior of discrete-time systems. A working knowledge of the z-transform is essential to the study of digital filters and systems. The definition of the z-transform's derivation from the Laplace transform of a discrete-time signal is presented first in this work. Seeing a system in terms of where the poles and zeros of the system transfer function are located in a complex plane is a helpful feature of the Laplace and z-transforms (Khanna, 2015).

2. The mathematical definition of the z-transform

The Laplace transform's discrete-time equivalent is called the z-transform. In this part, we use the Laplace transform of a discrete-time signal to obtain the z-transform. The Laplace transform L[f(z)], of a continuous-time function f(z), is obtained using the integral.

$$L[f(z)] = \int_0^\infty f(z)\, e^{-sz} dz = F(s) \quad \text{----- (i)}$$

In this context, the complex variable $= \alpha + i\beta$, s is a parameter which may be a real or complex and the interval from $z = 0\ to\ z = \infty$ accommodates the chances that the signal $f(z)$ may contain an impulse component [(Khanna, 2015), (Davies, 2012)].

Which is defined as $f(z) = \int_{-\infty}^{+\infty} F(s) e^{-sz} ds = L^{-1}[F(s)]$ ---------(ii)

Then $f(z)$ is called the inverse Laplace transform of F(s).

The selection of $-\infty$ ensures that L(s) is analytic (no singularities exist for s $> -\infty$). Eq. (i) can be used to sample the continuous-time input signal f(z) and obtain the z-transform. The Laplace transform Eq. (i) becomes for a sampled signal f(mTs), typically represented as f(m), suppose the sampling period $T_{s=1}$.

$$L[e^s] = \sum_{n=0}^{\infty} f[n]e^{-sn} - - - - - (iii)$$

By replacing the variable e^s in Eq. (iii) with the variable w, we derive the one-sided z-transform equation.

$$L[w] = \sum_{n=0}^{\infty} f[n]w^{-n} - - - - - (iv)$$

The two-sided z-transform is defined as

$$L[w] = \sum_{n=-\infty}^{\infty} f[n]w^{-n} - - - - - (v)$$

For a one-sided signal where $f(n) = 0$ for n < 0, Equations (iv) and (v) become identical.

The Fourier transform of a continuous time signal and the Laplace transform have a parallel relationship. The Laplace transform is a one-sided transformation where the integration starts at z $= 0$, while the Fourier transform is a two-sided transformation with the integration beginning at z $= -\infty$. For a one-sided signal that is zero for $z < 0$, the integration limits for both the Laplace transform and the Fourier transform are the same. The Laplace integral then becomes the Fourier integral if the frequency variable $i2\pi g$ is substituted for the variable s in the Laplace transform. Therefore, the Fourier transform is a particular instance of the Laplace transform for a one-sided signal, which corresponds to $s = i2\pi g$ and $\alpha = 0$[(Davies, 2012), (Debnath, 2016 ; Lathi, 2008)].

3. Between the Unit Circle and the z-Plane

The z-transform $w = re^{i\beta}$ and the frequency variables of the Laplace transform $z = \alpha + i\beta$ are complex variables with real and imaginary components that can be shown as a two-dimensional image. The Laplace transform's s-plane and the z-transform's z-plane. As also seen in Fig. (a), the horizontal α -axis provides the exponential rate of decay, or growth, of the amplitude of the complex sinusoid, while the vertical $i\omega$ -axis on the s-plane represents the frequency axis. As demonstrated.
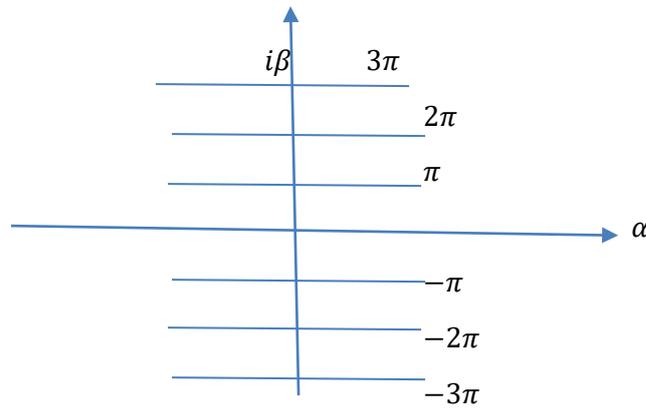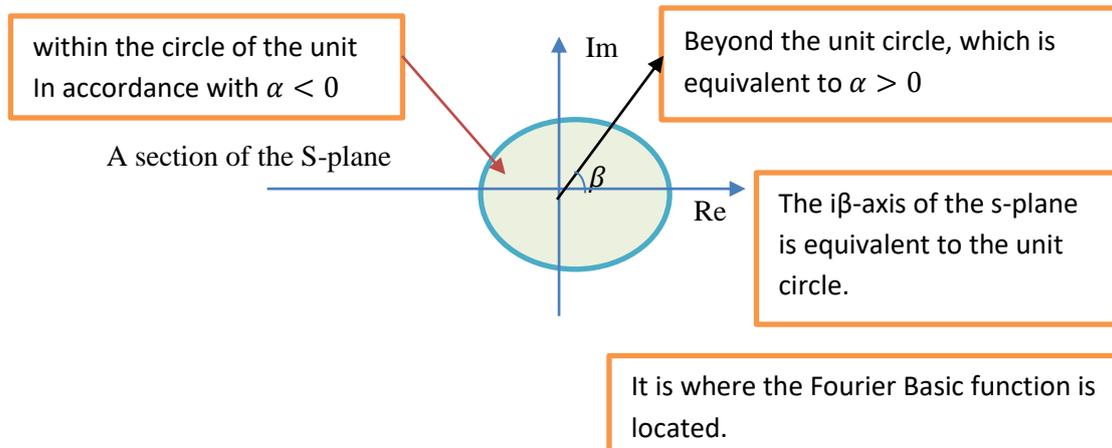
Fig. (a) – S-Plane

Fig. (b) – Z-Plane

Figure - Depiction of (a) the S-plane and (b) the Z-plane.

The s-plane and the signal's Laplace transform become recurring in relation to the $i\beta$ $-$axis when the signal is sampled in the time domain. The recurring horizontal dashed lines in Figure (a) serve as an illustration of this. The z-plane and its corresponding unit circle are examples of circular polar diagrams that are useful for representing periodic processes. In order to create a circle where the points π and $-\pi$ meet, visualize bending the $i\beta$ $-$axis of the s-plane of the sampled signal in Fig. (a) toward the left-hand side half of the s-plane. The final diagram is known as the z-plane, and the resulting circle is known as the unit circle. The region on the left side of the s-plane, where $\alpha < 0$ or $r = e^{\alpha} < 1$, corresponds to the area inside the unit circle. This represents the domain of stable causal signals and systems. The region to the right of the s-plane, where $\alpha < 0$ or $r = e^{\alpha} < 1$, corresponds to the exterior of the unit circle. This area represents unstable signals and systems. The $i\beta$-axis, where $\alpha = 0$ or $r = e^{\alpha} = 1$, corresponds to the unit circle in the complex plane. Therefore, for discrete-time

signals (Fig. b), the Cartesian coordinates utilized in the s-plane (Fig. a) are projected into a polar representation in the z-plane. (Diniz, 2010).

4.   The Region of Convergence (ROC)

Since the series is an infinite power series, the z-transform only appears for values of the variable z for which the series converges to a finite sum. The region of convergence (ROC) of L(w) refers to the collection of all values of w where L(w) yields a finite and calculable result (Proakis, 2001).

To determine the value of w that ensures the convergence of the series, we apply either the ratio test or the root test, which evaluate the behavior of a series involving complex numbers.

$\sum_{n=0}^{\infty} u_n$ with limit $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = U$

The series converges absolutely when U < 1, diverges when U > 1, and may or may not converge in other cases.

The root test indicates that if $\lim_{n\to\infty} \sqrt[n]{|u_n|} = U$

The series converges absolutely when U < 1, diverges when U > 1, and it is test failed when U = 1.

In general, the series converges absolutely provided that $\overline{\lim_{n\to\infty}} \sqrt[n]{|u_n|} < 1$, Where $\overline{\lim_{n\to\infty}}$ represents the maximum limit points of $\overline{\lim_{n\to\infty}} |f[nT]|^{\frac{1}{n}}$ and diverges if $\overline{\lim_{n\to\infty}} \sqrt[n]{|u_n|} > 1$.

By applying the root test to equation (iv), we derive the condition for convergence

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|f[nT]\, w^{-n}|} = \overline{\lim_{n\to\infty}} \sqrt[n]{|f[nT]|\, |w^{-1}|^n} < 1$$

$$|w| > \overline{\lim_{n\to\infty}} \sqrt[n]{|f[nT]|} = \Re$$

Here, The radius of convergence of the series is denoted by $\Re$.

Consequently, the series is centered at the origin and will converge completely for all points in the z-plane that are beyond the circle of radius $\Re$. This area is referred to as the region of convergence (ROC).

Example 1: Find the zone of convergence, or the z-transform. Regarding the signal

$f(n) = x^n u(n) = \begin{cases} x^n\ for\ n \geq 0 \\ 0\ for\ n < 0 \end{cases}$.

Solution: By definition $L[w] = \sum_{n=0}^{\infty} f[n]w^{-n}$

Using above, $L[w] = \sum_{n=0}^{\infty} x^n u(n)w^{-n}$
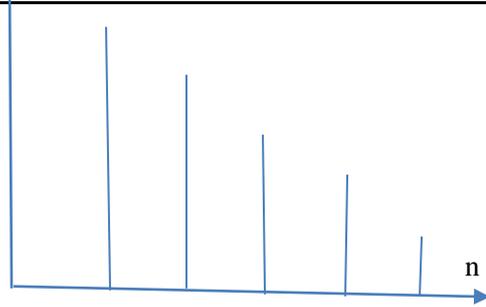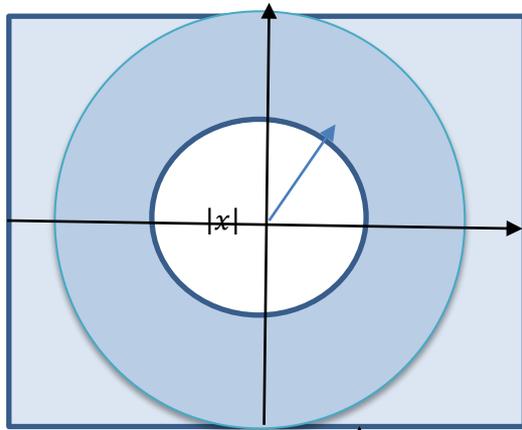
Since, $u(n) = 1$ for all $n \geq 0$.

$f(n)$

Fig. (c).

Or, $L[w] = \sum_{n=0}^{\infty}(xw^{-1})^n$

Or, $L[w] = \sum_{n=0}^{\infty}\left(\frac{x}{w}\right)^n$

Or, $L[w] = 1 + \frac{x}{w} + \left(\frac{x}{w}\right)^2 + \left(\frac{x}{w}\right)^3 + - - - -$

Im

Re

Fig. (d).

It can be useful to recall the following familiar geometric progression and its corresponding sum:

$1 + x + x^2 + x^3 + - - - = \frac{1}{1-x} \; if \; |x| < 1.$

Using this in $L[w] = 1 + \frac{x}{w} + \left(\frac{x}{w}\right)^2 + \left(\frac{x}{w}\right)^3 + - - - -$ we get

Or, $L[w] = \frac{1}{1-\frac{x}{w}}, \left|\frac{x}{w}\right| < 1$

$\qquad = \frac{w}{w-x}, |w| > 1$

Notice that $L(w)$ exists only for $|w| > |x|$. For $|w| > 1$, the sum in equation $L[w] = 1 + \frac{x}{w} + \left(\frac{x}{w}\right)^2 + \left(\frac{x}{w}\right)^3 + - - - -$ diverges, tending toward infinity.

Thus, the region of convergence (or existence) of $L(w)$ refers to the shaded area beyond the circle with a given radius, with the origin at its center in the z-plane, as illustrated in Fig. [a, b, c, d] (Adsad, 2015).

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