

A confirmation of the expected interval: Ab initio estimation technique of parsimonious Gompertz mortality parameters

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Abstract

The Gompertz law states a functional relationship on exponential scale between instantaneous intensity and age. The objective is to first estimate the model parameters by using mortality data and then confirm the interval of validity for the estimated parameters. The parsimonious model is implicitly expressed in terms of age and level of mortality while the force of mortality is the dependent variable. Current contributions in actuarial literature have made it tractable to obtain life span from the actuarial point of view, making the life table invaluable analytical tool for insurers. Mortality functions which have been developed recently possess sophisticated actuarial techniques with many parameters hence they are very complex to estimate numerically making it difficult to fit to mortality data. In order to overcome this problem, we need to employ numerical algebraic method to estimate the appropriate values of model parameters and which may enable us fit the function to mortality data. In this paper, the direct algebraic method offers simpler perspective of approximating mortality parameter and was decomposed into systems of algebraic equations. We observed that mortality C over all ages for males is lower than that of females while the initial mortality B for male is higher than that of female. The R-language software was employed in the computation. In view of actuarial benchmarks, our results confirm that the values

of B and C for both males and females lie within the expected interval $\frac{1}{10^6} < B < \frac{1}{10^3}$ and

$108 \times 10^{-2} < C < 112 \times 10^{-2}$. Furthermore, by reason of extra risk ζ , our results show that

$$\zeta \int_0^n \mu_{x+s} ds = e^{\beta + \zeta x + \zeta n} - e^{\beta + \zeta x}.$$

Keywords: Gompertz, intensity, mortality, interval of validity, extra risk

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Introduction

A life table is an actuarial tool summarizing the mortality experience of a cohort and it produces information on longevity and life expectancy. Despite its use in studying mortality trends, the life table can further be used to summarize any duration function such as duration of marriage, duration of sickness or duration of disability. Mortality table are mainly applied in life contingencies' computations to value life and pensions funds, however the procedures of constructing mortality table is quite involving taking time and resources to finish up. Within the time interval from the commencement to the completion, the set assumptions called the basis functions could have varied. The change could be due to innovations in health care delivery or outbreak of a pandemic like covid-19. The change in question could make the mortality table become irrelevant thus the risk could further render it unusable within that defined period. Mortality table remains potent drivers in life insurance contract product pricing and financial stability for life insurance liabilities. Policy regulators continue to express great concerns the level of deviation arising between the forecasted and the actual mortality because the deviation will result in cash flow distortions that may make life office financially unstable. The unexpected will definitely impact on underwriters' cash payment and reduced insurance business resulting in financial instability. Consequently, profit margins are affected because claims amount out go exceeds premium income collected. In order to compute appropriate premium, life offices need. We observe in Andreeski & Vasant (2008), that various functions were developed to estimate life tables but are really very complex to permit the technique of simple methods such as direct algebraic method, maximum likelihood or methods of moments to estimate the parameters. Following Higgins(2003); Canudas-Romo(2008); Missov, Lenart , Nemeth , Canudas-Romo & Vaupel (2015) and Chukwu & Ogunde (2016) hence some efforts are made to model the functional form which could estimate mortality data in an efficient manner. Various authors such as Strehler & Mildvan (1960) have proposed genetical theories to support the exponential increase in death rates as age increases. All of these theories are deeply rooted in Gompertz function formulated by Gompertz (1825) and modeled as $\mu(\tau) = Ke^{\delta\tau}$ where μ is the mortality intensity, δ and k are constants and τ is the age parameter. Other theories employing the Gompertz model is the Strehler & Mildvan theory. The Strehler & Mildvan theory falls in line with Gompertzian mortality kinetics but has a linear decay of physiological function at a rate consistent with observation. The theory can project the numerical inverse relation between Gompertz slopes and intercept which has been observed. The description of the various causes of death across age spectrum which leads to increase in mortality when using Gompertz model has proved helpful as a build-up of the theories outlined above. In view of Bongaarts (2005), Actuaries have adopted as a norm the functional relationships between age and mortality rate so that any deviation from this norm suggests that within any age spectrum, the population being treated has abnormal risk. Over the years, estimations of mortality tables have been constructed in connection with distinct functions other than the functions mentioned earlier. However, one of the most successful model formulated is the Gompertz-Makeham's model. This model results in higher degree of accuracy approximations for mortality data other than the ones explained earlier. The Gompertz-Makeham model is wholly built on the Gompertz mortality function. In Mahdi & Gupta (2012), Cohen, Bohk-Ewald & Rau (2018), this function is usually applied to model human mortality and to compute life tables motivating other estimation methods being suggested for this model with respect to complete and censored mortality data. The choice of approximation technique relies wholly on the area of interest. As observed in Mahdi & Gupta (2012), it is most probable that the technique of approximations is differently appealing to

distinct users. Following Melnikov & Romaniuk (2006) and Tai, Noymer (2017), this new variation of Gompertz' distribution also has aroused much interest in actuarial literature. In Willemse & Koppelaar (2000), the quintile function of Gompertz-Makeham has been associated with a closed form equation by applying the Lambert W function. Mortality at extreme ages seems to deviate from the Gompertz's model, hence the logistic model has been proposed to take care of this departure to fit human mortality. The approximations of mortality rate at extreme ages are hard to compute since only a few lives survive to these ages. Moreover, mortality data at extreme ages are rare and subject to age exaggeration. The deterministic demographic approximations of mortality based on period data, experience well-known denominator problems. In order to calculate better estimates of mortality at extreme ages, mortality data are combined together for different calendar periods. The distribution of mortality by single year shows few numbers of lives surviving to the end of the life table making approximations of mortality at extreme ages inadequate. In Zemleni (2006) and Mahdi & Gupta (2012), the Gompertz's model is applied for modeling human lifetimes in the context of life insurance. In a research conducted to test the adequacy of Taylor's law on Gompertz's, Makeham's and Siler's models, Cohen, Bohk-Ewald & Rau (2018) extended the senescent mortality of Gompertz to 100 years

Preliminary Survey of Standard Actuarial Functions

According to McNown & Rogers (1992), the mortality rate is a measure of the number of deaths in general due to aging or through a specified cause in a defined population, scaled to the size of that population, per unit of time. Mortality is measured because it is useful to public health authorities. Vital statistics systems which is compulsory in many economies, record certain information contained on every death, such as name, age at deaths, and cause of death, then add the number of deaths periodically to compute mortality rates. Mortality data remain good source of information concerning the health status of surviving communities. It generates a summary of the health risk profile, identifies continuous and non-transient patterns of risk profile in certain communities and shows trend patterns in particular causes of death.

For a live age x , its future lifetime is $T_x = X - x$. For a newborn, $x = 0$,

so that we have $T_0 = X$ (1)

$$S_x(s) = \Pr[T_0 > x + s | T_0 > x] = \frac{S_0(x+s)}{S_0(x)} = {}_s p_x = 1 - {}_s q_x \quad (2)$$

$$F_x(s) = \Pr[T_0 \leq x + s | T_0 > x] = \frac{S_0(x) - S_0(x+s)}{S_0(x)} = {}_s q_x \quad (3)$$

$$f_x(s) = \frac{dF_x(s)}{ds} = \frac{dS_x(s)}{ds} = \frac{f_0(x+s)}{S_0(x)} \quad (4)$$

p_x refers to the probability that (x) survives for another year.

$q_x = 1 - p_x$, on the other hand, refers to the probability that (x) dies within one year.

Following Bowers *et al.* (1997); Hudec (2017); Ogungbenle & Ogungbenle (2020); Ogungbenle

$$\& Adeyele(2020) \mu_x(s) = \frac{f_x(s)}{S_x(s)} = \frac{f_0(x+s)}{S_0(x)} \cdot \frac{S_0(x)}{S_0(x+s)} = \frac{f_0(x+s)}{S_0(x+s)} = \mu_{x+s} \quad (5)$$

$$f_x(s) = {}_s p_x \times \mu_{x+s} = \prod_{t=0}^{s-1} (1 - q_{x+t}) \times \mu_{x+s} \text{ for } s \text{ integral} \tag{6}$$

The probability that (x) will survive for s years and die within the next u years is denoted by ${}_{s|v} q_x$.

This is equivalent to the probability that (x) will die between the ages of $x + s$ and $x + s + v$

This can be computed in several ways:

$${}_{s|v} q_x = \Pr[t > T_x \leq s + v] \tag{7}$$

$$= \Pr[T_x \leq s + v] - \Pr[T_x < s] \tag{8}$$

$$= {}_{s+v} q_x - {}_s q_x \tag{9}$$

$$= {}_s p_x - {}_{s+v} p_x \tag{10}$$

$$= {}_s p_x \times {}_v q_{x+s} = {}_v q_{x+s} \prod_{t=0}^{s-1} (1 - q_{x+t}) \tag{11}$$

We see that $F_x(s) = \int_0^s f_x(\tau) d\tau$ (12)

In actuarial notation, ${}_s q_x = \int_0^s {}_\tau p_x \mu_{x+\tau} d\tau$ (13)

We can generalize this to ${}_{t|v} q_x = \int_t^{t+v} {}_\tau p_x \mu_{x+\tau} d\tau$ (14)

Curtate future lifetime of (x) is the number of future years completed by a life aged (x) prior to death.

$K_x = [T_x]$, the greatest integer part of T_x

Its probability mass function is

$$\Pr[K_x = k] = \Pr[k \leq T_x < k + 1] = \Pr[k < T_x \leq k + 1] \tag{15}$$

$$= S_x(k) - S_x(k + 1) = {}_{k+1} q_x - {}_k q_x = {}_{k|} q_x, \tag{16}$$

for $k = 0, 1, 2, 3, \dots$

Its distribution function is $\Pr[K_x \leq k] = \sum_{h=0}^k {}_{h|} q_x = {}_{k+1} q_x$ (17)

The expected value of T_x is called the complete expectation of life:

$$e_x = E[T_x] = \int_0^\infty s f_x(s) ds = \int_0^\infty s {}_s p_x \mu_{x+s} ds \tag{18}$$

$$e_x = \int_0^\infty -s \frac{\partial_s p_x}{\partial x} ds = - \left[s \frac{\partial_s p_x}{\partial x} \right]_0^\infty + \int_0^\infty {}_s p_x ds \tag{19}$$

$$e_x = \int_0^\infty {}_s p_x ds \tag{20}$$

The expected value of K_x is called the curtate expectation of life:

$$e_x = \sum_{k=1}^\infty k p_x = (p_x) E[K_x / T_x \geq (p_x + q_x)] + \tag{21}$$

$$(p_x) E[K_x / T_x < (p_x + q_x)] = (1 - q_x)(1 + e_{x+1})$$

Let ρ and λ be real constants and assume that the survival function of a newborn is exponentially defined as $\lambda S_X(s) = -\rho(e^{\lambda s} - 1)$ and cumulative hazard function be

$$Z(s) = \log_e \left[\frac{1}{S_X(s)} \right] = \log_e [S_X(s)]^{-1} = -\log_e [S_X(s)] \tag{22}$$

Let θ be the hazard function intensity, then $\theta(s) = \frac{dZ(s)}{ds} = -\frac{\frac{dS_X(s)}{ds}}{S_X(s)}$ (23)

$$Z(s) = \rho \frac{e^{\lambda s} - 1}{\lambda}, \theta(s) = \rho e^{\lambda s} \tag{24}$$

From equation (22), $S_X(s) = e^{-Z(s)} \Rightarrow F_X(s) = 1 - e^{-Z(s)}$

$$\frac{dF_X(s)}{ds} = f_X(s) = \frac{d(1 - e^{-Z(s)})}{ds} = Z'(s)e^{-Z(s)} = \theta(s)e^{-Z(s)} \tag{25}$$

$$\frac{dF(s)}{ds} = \rho e^{\lambda s} e^{-\left(\rho \frac{e^{\lambda s} - 1}{\lambda}\right)} = \rho e^{\lambda s - \left(\rho \frac{e^{\lambda s} - 1}{\lambda}\right)} \tag{26}$$

$$\frac{d^2 F(s)}{ds^2} = \frac{df(s)}{ds} = \theta'(s)e^{-Z(s)} - \theta(s)e^{-Z(s)}Z'(s) \tag{27}$$

$$\frac{d^2 F(s)}{ds^2} = \theta'(s)e^{-Z(s)} - [\theta(s)]^2 e^{-Z(s)} = \tag{28}$$

$$\left\{ \theta'(s) - [\theta(s)]^2 \right\} e^{-Z(s)}$$

$$\frac{d^2 F(s)}{ds^2} = \left\{ \rho \lambda e^{\lambda s} - [\rho e^{\lambda s}]^2 \right\} e^{-Z(s)} = \left\{ \rho \lambda e^{\lambda s} - \rho^2 e^{2\lambda s} \right\} e^{-Z(s)} \tag{29}$$

$$\frac{d^3 F(s)}{ds^3} = \left\{ \rho \lambda^2 e^{\lambda s} - 2\lambda \rho^2 e^{2\lambda s} \right\} e^{-Z(s)} - Z'(s) \left\{ \rho \lambda e^{\lambda s} - \rho^2 e^{2\lambda s} \right\} e^{-Z(s)} \tag{29a}$$

$$\frac{d^3 F(s)}{ds^3} = e^{-Z(s)} \left[\left\{ \rho \lambda^2 e^{\lambda s} - 2\lambda \rho^2 e^{2\lambda s} \right\} - \theta(s) \left\{ \rho \lambda e^{\lambda s} - \rho^2 e^{2\lambda s} \right\} \right] \tag{29b}$$

$$\frac{d^3 F(s)}{ds^3} = e^{-Z(s)} \left[\rho \lambda^2 e^{\lambda s} - 2\lambda \rho^2 e^{2\lambda s} - \rho^2 \lambda e^{2\lambda s} + \rho^3 e^{3\lambda s} \right] = \tag{29c}$$

$$e^{-Z(s)} \left[\rho \lambda^2 e^{\lambda s} - 3\lambda \rho^2 e^{2\lambda s} + \rho^3 e^{3\lambda s} \right]$$

From equation (29), If $\left\{ \rho \lambda e^{\lambda s} - \rho^2 e^{2\lambda s} \right\} e^{-Z(s)} = 0$, $\rho \lambda e^{\lambda s} - \rho^2 e^{2\lambda s} = 0 \Rightarrow \rho \lambda e^{\lambda s} = \rho^2 e^{2\lambda s}$ (30)

Let $\zeta = \rho e^{\lambda s} \Rightarrow \lambda \zeta - \zeta^2 = 0 \Rightarrow \zeta(\zeta - \lambda) = 0 \Rightarrow \zeta = \lambda, \zeta = 0$ (31)

Thus there are two points of inflexion

$$\rho e^{\lambda s} = 0 \Rightarrow \rho = 0 \quad (32)$$

$$\rho e^{\lambda s} = \lambda \Rightarrow \rho + \rho \lambda s = \lambda \Rightarrow s = \frac{1}{\rho \lambda} (\lambda - \rho) \quad (33)$$

From equation (33), ρ cannot be zero

2.1 Procedures For Obtaining Estimated Probability Of Survival

$$\text{We define, } l_{x+s} = l_x + s\Delta l_x + \frac{s(s-1)}{2!} \Delta^2 l_{x-1} \quad (34)$$

where $\Delta l_x = l_{x+1} - l_x$ is the forward differencing

$$\mu_x = \frac{-1}{l_x} \left(\frac{dl_{x+s}}{ds} \right)_{s=0} = \frac{-1}{l_x} (\Delta l_x - 0.5\Delta(\Delta l_{x-1})) = 0.5 \frac{l_{x-1}}{l_x} - 0.5({}_1P_x) \quad (35)$$

Using a Gauss forward formula to order 4, and following Ogungbenle & Adeyele (2020), we

$$\text{have } l_{x+s} = l_x + s\Delta l_x + \frac{(s^2 - s)}{2} \Delta^2 l_{x-1} + \frac{(s^3 - s)}{6} \Delta^3 l_{x-1} + \frac{(s^3 - s)(s-2)}{24} \Delta^4 l_{x-2} \quad (36)$$

$$\mu_x = \frac{-1}{l_x} \left(\frac{dl_{x+s}}{ds} \right)_{s=0} = \frac{-1}{l_x} \left(\Delta l_x - 0.5\Delta(\Delta l_{x-1}) - \frac{1}{6} \Delta(\Delta^2 l_{x-1}) + \frac{1}{12} \Delta(\Delta^3 l_{x-2}) \right) \quad (37)$$

hence

$$12l_x \mu_x = 8(l_{x-1} - l_{x+1}) - (l_{x-2} - l_{x+2}) \quad (38)$$

$$\text{Now, } dl_x = -l_x \mu_x dx \quad (38a)$$

$$[l_{x+t}]_0^m = -\int_0^m (\mu_{x+t}) l_{x+t} dt \quad (38b)$$

$$l_{x+m} - l_x = -\int_0^m (\mu_{x+t}) l_{x+t} dt \quad (38c)$$

$$l_x - l_{x+m} = \int_0^m (\mu_{x+t}) l_{x+t} dt \Rightarrow {}_m d_x = \int_0^m (\mu_{x+t}) l_{x+t} dt \quad (38d)$$

$$\text{Using equation (38), } {}_m q_x = \int_0^m \left(\frac{8(l_{x+s-1} - l_{x+s+1}) - (l_{x+s-2} - l_{x+s+2})}{12l_{x+s}} \right) ({}_t P_{x+s}) ds \quad (38e)$$

$${}_m q_x = \int_0^m \left(\frac{8(l_{x+s-1} - l_{x+s+1}) - (l_{x+s-2} - l_{x+s+2})}{12} \right) l_{x+s+t} ds \quad (38f)$$

$$q_0 = \int_0^1 \left(\frac{8(l_{s-1} - l_{s+1}) - (l_{s-2} - l_{s+2})}{12} \right) l_{s+t} ds \tag{38g}$$

thus q_0 is independent of age x at time s

From equation (38d), $q_x = \int_0^1 (\mu_{x+t})_t P_x dt \Rightarrow l_x q_x = \int_0^1 (\mu_{x+t}) l_{x+t} dt$

$$\frac{d}{dt} (\mu_{x+t}) l_{x+t} \geq 0 \Rightarrow$$

$$\left(\frac{d}{dt} \mu_{x+t} \right) l_{x+t} + (\mu_{x+t}) \frac{d}{dt} l_{x+t} = (\mu_{x+t}^2) l_{x+t} - (\mu_{x+t}^2) l_{x+t} = 0$$

hence $(\mu_{x+t}) l_{x+t}$ is non-decreasing, and $(\mu_{x+t}) l_{x+t} \geq 0$

then within the interval $0 \leq t \leq 1$, $l_x \mu_x < l_x q_x \Rightarrow \mu_x < q_x$ in the life table

Theorem: If ${}_t q_x$ is a continuous function, then it is possible to obtain the minimum where

$${}_t P_x = \frac{[8(l_{x+t-1} - l_{x+t+1}) - (l_{x+t-2} - l_{x+t+2})]}{[8(l_{x-1} - l_{x+1}) - (l_{x-2} - l_{x+2})]} \text{ and } \mu_{x+1} = \mu_x \tag{38f}$$

Proof

Note that by definition, $\mu_x = \lim_{\Delta t \rightarrow 0^+} \frac{\Delta({}_t q_x)}{\Delta t} = \lim_{\Delta t \rightarrow 0^+} \frac{\Pr(T(x) < \Delta t)}{\Delta t}$

$$\frac{d}{dx} (1 - {}_t P_x) = {}_t P_x (\mu_{x+t} - \mu_x) = \frac{d}{dx} ({}_t q_x) \tag{38h}$$

$${}_t P_x (\mu_{x+t} - \mu_x) = 0 \Rightarrow (\mu_{x+t} - \mu_x) = 0, \quad {}_t P_x \neq 0 \tag{38i}$$

$$\mu_{x+t} = \mu_x, \tag{38j}$$

$$\frac{1}{12l_{x+t}} [8(l_{x+t-1} - l_{x+t+1}) - (l_{x+t-2} - l_{x+t+2})] = \frac{1}{12l_x} [8(l_{x-1} - l_{x+1}) - (l_{x-2} - l_{x+2})] \tag{38k}$$

$$[8(l_{x+t-1} - l_{x+t+1}) - (l_{x+t-2} - l_{x+t+2})] = \frac{l_{x+t}}{l_x} [8(l_{x-1} - l_{x+1}) - (l_{x-2} - l_{x+2})] \tag{38l}$$

$$[8(l_{x+t-1} - l_{x+t+1}) - (l_{x+t-2} - l_{x+t+2})] = {}_t P_x [8(l_{x-1} - l_{x+1}) - (l_{x-2} - l_{x+2})] \tag{38m}$$

$${}_t P_x = \frac{[8(l_{x+t-1} - l_{x+t+1}) - (l_{x+t-2} - l_{x+t+2})]}{[8(l_{x-1} - l_{x+1}) - (l_{x-2} - l_{x+2})]} \tag{38n}$$

since $t = 0$ is not possible, consequently

so at $t = 1$

$${}_1P_x = \frac{[8(l_x - l_{x+2}) - (l_{x-1} - l_{x+3})]}{[8(l_{x-1} - l_{x+1}) - (l_{x-2} - l_{x+2})]} \text{ and} \tag{38p}$$

$\mu_{x+1} = \mu_x$ hence, ${}_1q_x$ attains its minimum

If ${}_tP_x$ were continuous in the interval $0 \leq t \leq 1$, then $l_{x+t} = (1-t)l_x + tl_{x+1}$ for integral x , then (38n) becomes

$${}_tP_x = \frac{\{[8(2-t)l_x + (t-1)l_{x+1} + tl_x + (t+1)l_{x+1}] - [(3-t)l_x + (t-2)l_{x+1} - (-1-t)l_x + (t+2)l_{x+1}]\}}{[8(l_{x-1} - l_{x+1}) - (l_{x-2} - l_{x+2})]} \tag{39q}$$

$${}_tP_x = \frac{\{(16-8t)l_x + (t-1)l_{x+1} + tl_x + (t+1)l_{x+1} - (3-t)l_x - (t-2)l_{x+1} - (1+t)l_x - (t+2)l_{x+1}\}}{[8(l_{x-1} - l_{x+1}) - (l_{x-2} - l_{x+2})]} \tag{39r}$$

$${}_tP_x = \frac{\{l_x \langle 16-8t+t-3+t-1-t \rangle + l_{x+1} \langle t-1+t+1-t+2-t-2 \rangle\}}{[8(l_{x-1} - l_{x+1}) - (l_{x-2} - l_{x+2})]} \tag{39s}$$

$${}_tP_x = \frac{\{ \langle 8-7t \rangle l_x \}}{[8(l_{x-1} - l_{x+1}) - (l_{x-2} - l_{x+2})]} \tag{39t}$$

At $t = \frac{1}{2}$, ${}_tP_x = \frac{l_x}{16(l_{x-1} - l_{x+1}) - 2(l_{x-2} - l_{x+2})}$ (39u)

Material and Methods

Gompertz Model

Bowers *et al.* (1997) define the force of mortality as Gompertz apply the exponential function for the instantaneous mortality intensity. For life offices, the model has implications for the parameters and also has minimum number of parameters for obtaining mortality trend over time period. This parsimony accounts for the reason why Gompertz law is justified in mortality. It is observed in literature that within a known collection of mortality models, modest descriptions with respect to their number of parameters are much more preferred to the sophisticated mortality models. Parsimony hence refers to the cardinality of efficient model parameters. Consequently, the Gompertz model offers a good trade-off between a simple numerical computational implementation and efficient analytic process. Gompertz is intuitive and hence reduces the risk of an incongruous implementation. The mortality intensity is defined as follows

In Gompertz (1825), $\mu_x = BC^x = e^{\beta+\zeta x}$, $x > 0$ and subject to: (39)

$$\begin{cases} C > 1 \\ B \neq 1 \end{cases} \tag{39a}$$

The constant B is the level of the mortality intensity of a newborn while the constant C describes the rate of demographic ageing. Thus, the death probability increases at a constant exponential rate when age increases in the equation implying exponential ageing. In the logarithmic scale $\ln \mu_x = \beta + \zeta x$ translating to the fact that logarithmic function of Gompertz's law is linearly increasing B is the starting point of parameter and C is the rate of mortality over ages while x is the initial age of analysis

$$\int_0^x \mu_s ds = \int_0^x BC^s ds \tag{40}$$

$$\int_0^x \mu_s ds = \left[\frac{BC^s}{\log_e C} \right]_0^x \Rightarrow \frac{BC^x}{\log_e C} - \frac{B}{\log_e C} \tag{41}$$

$$\int_0^x \mu_s ds = -(C^x - 1) \log_e g, \text{ where, } \log_e g = \frac{-B}{\log_e C} \tag{42}$$

$$\int_0^x \mu_s ds = -\log_e g C^{x-1} \tag{43}$$

$$l_x = l_0 e^{-\int_0^x \mu_s ds} = l_0 e^{\log_e g C^{x-1}} = l_0 g^{C^x-1} \tag{44}$$

$$l_x = \frac{l_0}{g} g^{C^x} = kg^{C^x}, k = \frac{l_0}{g} \tag{45}$$

$${}_s p_x = \frac{kg^{C^{x+1}}}{kg^{C^x}} = {}_s p_x = g^{C^x(C'-1)} \tag{46}$$

Again, as $\lim_{x \rightarrow \omega} {}_t P_x \rightarrow 0$

$$\int_0^\omega {}_t f_T(t) dx = \int_0^\omega t \mu(x+t) {}_t P_x dx \tag{47}$$

$$\int_0^\omega {}_t f_T(t) dt = \int_0^\omega t \frac{\partial({}_t P_x)}{\partial t} dt = {}_t P_x \Big|_{t=0}^{t=\omega} - \int_0^\omega \frac{\partial({}_t P_x)}{\partial t} dt \tag{48}$$

$$\int_0^\omega {}_t f_T(t) dt = -\int_0^\omega \frac{\partial({}_t P_x)}{\partial t} dt = \int_0^\omega {}_t P_x dt \tag{49}$$

$$\int_0^\omega {}_t f_T(t) dt = -\int_0^\omega \frac{\partial({}_t P_x)}{\partial t} dt = \int_0^\omega g^{C^x(C'-1)} dt \tag{50}$$

Data Presentation and Analysis

Life offices usually would like to consider the level at which a live commences a career in order to permit them to assess the salary structure of an insured (x) and to make it more rewarding when (x) are dealing with the life office. Life offices usually make assumption to set the age where mortality data is fitted by a model. Again, many lives who start a career obtain university

education or equivalents and thereafter assume an office around 20 years. Consequently, as we consider the career of individuals, it seems appropriate to collect data from age 20 .

Using the formulae that are developed above, we construct the following results for male and female using R-language. For the purpose of this study, the data used in this study came mainly from the mortality of the population of England and Wales during the years 1990, 1991 and 1992.

For Males

$$98496 = kg^{c^{20}} \quad (51)$$

$$96500 = kg^{c^{40}} \quad (52)$$

$$86714 = kg^{c^{60}} \quad (53)$$

Divide equation (52) by (51) and equation (53) by (51)

$$\frac{96500}{98496} = \frac{kg^{c^{40}}}{kg^{c^{20}}} \quad (54)$$

$$0.9797352177 = g^{c^{20}(c^{20}-1)} \quad (55)$$

$$\frac{86714}{96500} = \frac{kg^{c^{60}}}{kg^{c^{40}}} \quad (56)$$

$$0.8985906736 = g^{c^{40}(c^{20}-1)} \quad (57)$$

From equation (55)

$$\log_e 0.9797352177 = c^{20}(c^{20} - 1)\log_e g \quad (58)$$

From equation (57)

$$\log_e 0.8985906736 = c^{40}(c^{20} - 1)\log_e g \quad (59)$$

Divide (59) by (58)

$$\frac{\log_e 0.8985906736}{\log_e 0.9797352177} = \frac{c^{40}(c^{20} - 1)\log_e g}{c^{20}(c^{20} - 1)\log_e g} \quad (60)$$

$$5.222880263 = c^{20} \quad (61)$$

$$c = 1.086164248 \quad (62)$$

From equation (53) and (52)

$$\frac{86714}{96500} = \frac{kg^{(1.086164248)^{60}}}{kg^{(1.086164248)^{40}}} \quad (63)$$

$$0.8985906736 = 115.1937482g \quad (64)$$

$$\log_e 0.8985906736 = 115.1937482 \log_e g \quad (65)$$

$$\frac{\log_e 0.8985906736}{115.1937482} = \log_e g \quad (66)$$

$$-4.031303257 \times 10^{-4} = \log_e g \quad (67)$$

$$g = e^{-0.0004031303257} \quad (68)$$

$$g = 0.9995969509 \quad (69)$$

From equation (53)

$$86714 = kg^{c^{60}} \quad (70)$$

$$86714 = k(0.9995969509)^{142.4722266} \quad (71)$$

$$86714 = 0.9441833755k \quad (72)$$

$$k = 91840.21055 \quad (73)$$

$$\log_e g = \frac{-B}{\log_e C} \quad (74)$$

$$-B = \log_e g \times \log_e C \quad (75)$$

$$-B = \log_e 0.9995969509 \times \log_e 1.086164248 \quad (76)$$

$$-B = -6.284487294 \times 10^{-6} \quad (77)$$

$$B = 0.000006284487294 \quad (78)$$

$$\mu_x = Bc^x \quad (79)$$

$$\mu_x = 0.000006284487297(1.086164248)^x \quad (80)$$

$$l_x = (91840.210546049)0.9995969509^{(1.086164248)^x} \quad (81)$$

$$d_x = (91840.210546049)0.9995969509^{(1.086164248)^x} - (91840.210546049)0.9995969509^{(1.086164248)^{x+1}} \quad (82)$$

For Females

$$98957 = kg^{c^{20}} \quad (83)$$

$$97952 = kg^{c^{40}} \quad (84)$$

$$91732 = kg^{c^{60}} \quad (85)$$

Divide equation (84) by (83) and (85) by (84)

$$\frac{97952}{98957} = \frac{kg^{c^{40}}}{kg^{c^{20}}} \quad (86)$$

$$0.9898440737 = g^{c^{20}(c^{20}-1)} \quad (87)$$

$$\frac{91732}{97952} = \frac{kg^{c^{60}}}{kg^{c^{40}}} \quad (88)$$

$$0.93649951 = g^{c^{40}(c^{20}-1)} \quad (89)$$

From equation (87)

$$\log_e 0.9898440737 = c^{20}(c^{20} - 1)\log_e g \quad (90)$$

From equation (89)

$$\log_e 0.93649951 = c^{40}(c^{20} - 1)\log_e g \quad (91)$$

Divide (91) by (92)

$$\frac{\log_e 0.93649951}{\log_e 0.9898440737} = \frac{c^{40}(c^{20} - 1)\log_e g}{c^{20}(c^{20} - 1)\log_e g} \quad (92)$$

$$6.427042235 = c^{20} \quad (93)$$

$$c = 1.097489964 \quad (94)$$

From equation (85) and (84)

$$\frac{91732}{97952} = \frac{kg^{(1.097489964)^{60}}}{kg^{(1.097489964)^{40}}} \quad (95)$$

$$0.93649951 = 224.1741328g \quad (96)$$

$$\log_e 0.93649951 = 224.1741328\log_e g \quad (97)$$

$$\frac{\log_e 0.93649951}{224.1741328} = \log_e g \quad (98)$$

$$-1.27099613 \times 10^{-4} = \log_e g \quad (99)$$

$$g = e^{-0.000127099613} \quad (100)$$

$$g = 0.9998729085 \quad (101)$$

From equation (85)

$$91732 = kg^{c^{60}} \quad (102)$$

$$91732 = k(0.9998729085)^{265.4810041} \quad (103)$$

$$91732 = 0.9668204064k \quad (104)$$

$$k = 94880.08259 \quad (105)$$

$$\log_e g = \frac{-B}{\log_e c} \quad (106)$$

$$-B = \log_e g \times \log_e c \quad (107)$$

$$-B = \log_e 0.9998729085 \times \log_e 1.097489964 \quad (108)$$

$$-B = -0.00005519864486 \times 0.04040055753 \quad (109)$$

$$-B = -2.230056027 \times 10^{-6} \quad (110)$$

$$B = 0.000002230056027 \quad (111)$$

$$\mu_x = Bc^x \quad (112)$$

$$\mu_x = 0.000002230056027(1.097489964)^x \quad (113)$$

$$l_x = (94880.08259)0.9998729085^{(1.097489964)^x} \quad (114)$$

$$d_x = (94880.08259)0.9998729085^{(1.097489964)^x} \quad (115)$$

$$-(94880.08259)0.9998729085^{(1.097489964)^{x+1}}$$

Discussion of Result

The constructed empirical curves are the approximates of the corresponding theoretical functions. From our results, we observed that mortality over all ages(C-value) for males is lower than that of females while the initial mortality(B-value) for male is higher than that of female. Subject to the inequalities in (39a) and comparing with standard results in Bowers *et al.* (1997), our results strictly satisfy the constraints that the rate of demographic ageing $C > 1$ and initial mortality $B > 0$. Furthermore, based on our results the values of B and C for both males and

females lie within the expected interval $\frac{1}{10^6} < B < \frac{1}{10^3}$ and $108 \times 10^{-2} < C < 112 \times 10^{-2}$. (116)

From equations (80) and (81) above, the male probability density function is given by

$$f_{T(x)}(t) = 0.000006284487297(1.086164248)^{x+t} \times \frac{(91840.210546049)0.9995969509^{(1.086164248)^{x+t}}}{(91840.210546049)0.9995969509^{(1.086164248)^x}} \tag{117}$$

$$f_{T(x)}(t) = 0.000006284487297(1.086164248)^{x+t} \times \frac{0.9995969509^{(1.086164248)^{x+t}}}{0.9995969509^{(1.086164248)^x}} \tag{118}$$

while from equations (112) and (113) the female density is

$$f_{T(x)}(t) = 0.000002230056027(1.097489964)^{x+t} \times \frac{(94880.08259)0.9998729085^{(1.097489964)^{x+t}}}{(94880.08259)0.9998729085^{(1.097489964)^x}} \tag{119}$$

$$f_{T(x)}(t) = 0.000002230056027(1.097489964)^{x+t} \times \frac{0.9998729085^{(1.097489964)^{x+t}}}{0.9998729085^{(1.097489964)^x}} \tag{120}$$

$\mu_0^{FEMALE} = 0.000002230056027$ for female and the estimates of $\mu_0^{MALE} = 0.000006284487297$ for male although mortality changes at initial stages of life within age interval $0 \leq x < 1$ and as a result of the variation in this interval, obtaining values for μ_0 is usually not unique. Furthermore, it is also possible that the force of mortality at very old age is greater than 1 due to the fact that force of mortality is not a probability and consequently one expects that its value may exceed 1 towards the end of the mortality table.

However, given an extra risk ξ involving a constant sum to the mortality intensity

$$\text{then } \mu_x^E = \mu_x + \xi = e^{\beta+\zeta x} + \xi \tag{121}$$

$$\text{The force of interest } \delta = \log_e(1+i) \tag{122}$$

$$\log_e({}_n P_x) = -\int_0^n \mu_{x+s} ds. \tag{123}$$

Therefore,

$$\log_e({}_n P_x^E) = -\int_0^n (\mu_{x+s} + \xi) ds = -\int_0^n \mu_{x+s} ds - \int_0^n \xi ds \tag{124}$$

$$\log_e({}_n P_x^E) = -\int_0^n \mu_{x+s} ds - n\xi = \log_e({}_n P_x) - n\xi \tag{125}$$

$$\log_e({}_n P_x^E) - \log_e({}_n P_x) = -n\xi \tag{126}$$

$$\log_e \frac{{}_n P_x^E}{{}_n P_x} = -n\xi \tag{127}$$

$$\frac{{}_n P_x^E}{{}_n P_x} e^{-\delta n} = e^{-n(\xi+\delta)} \Rightarrow ({}_n P_x^E) e^{-\delta n} = ({}_n P_x) e^{-n(\xi+\delta)} = ({}_n P_x) (v^E)^n \tag{128}$$

$$({}_n P_x) e^{-n(\xi+\delta)} = ({}_n P_x) e^{-n\delta^E} \Rightarrow e^{-n\delta^E} = e^{-n(\xi+\delta)} \tag{129}$$

By definition, $\delta : J \rightarrow R$ is a continuous function on an interval $J \subset R^+$. If \exists a piecewise differentiable continuous function $A : J \rightarrow R$ such that $\frac{dA}{ds} = \delta(s)A(s)$, then

$$A(s) = e^{\int_c^s \delta(\zeta) d\zeta} A(c) \text{ and since } v^E = e^{-\delta^E}$$

we have $\delta^E = \xi + \delta$, hence the force of interest has been increased correspondingly. Putting

$$(121) \text{ in } (124), \text{ we have, } \log_e ({}_n P_x^E) = -\int_0^n (e^{\beta+\zeta x+\zeta s}) ds - \int_0^n \xi ds \tag{130}$$

$$\log_e ({}_n P_x^E) = \left[\frac{e^{\beta+\zeta x+\zeta s}}{\zeta} \right]_0^n - n\xi = \frac{e^{\beta+\zeta x}}{\zeta} - \frac{e^{\beta+\zeta x+\zeta n}}{\zeta} - n\xi \tag{131}$$

$$\text{and by } (125), \log_e ({}_n P_x) = \frac{e^{\beta+\zeta x}}{\zeta} - \frac{e^{\beta+\zeta x+\zeta n}}{\zeta} \tag{132}.$$

$$\text{Hence using } (123), \zeta \int_0^n \mu_{x+s} ds = e^{\beta+\zeta x+\zeta n} - e^{\beta+\zeta x} \tag{133}$$

Conclusion

The paper is motivated to study the actuarial properties of the Gompertz force of mortality and to provide explicit actuarial expressions to estimate model unknown parameters. Although our intention is to confirm the interval of validity within which the parameters fall, survival probabilities was also estimated at off grid points and the effect of an additional given extra risk ξ involving a constant sum to the mortality intensity was also investigated to capture the relationship between the extra risk and the interest rate intensity.

This study would assist life offices in evaluating actuarial risks connected with death probabilities as we applied simple numerical algebraic procedure to compute the approximate values of the model parameters so as to fit the model to interval of validity. Life offices will understand how to apply an efficient method of analyzing mortality data from a defined population of insured thereby giving an inference into the Gompertz distribution. It is therefore necessary for life offices to allow for upward trend in life expectancy and decrease in mortality rates. The underlying principle of life insurance necessitates that premium advised to the insured should fall in line with his risk profile. An insured’s age structure seems to be the most influential factor in appraising risk profile and hence our construction can be used to compute mortality table and hence life annuity & life assurance products. Furthermore, sex is another important factor since it has been observed from

our results that male have higher level of mortality rate than female. Because of genetical and behavioral differences, life offices should advise measurable premiums to male and female depending on sex. Life offices may be motivated to assess policyholder's risk profile accurately so as to permit competitive premiums as a result of market pressures. This is usually done by recognizing risk rating determinants which are cost effective in risk assessment process through actuarial techniques relevant to death probabilities calculations. It is clear that, the accuracy of the age-specific mortality rates is influenced by the accuracy of the age distribution of population and deaths.

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