3*n*+1 Problem and its Dynamics

[1]Bishnu Hari Subedi and [2]Ajaya Singh

[1, 2] *Central Department of Mathematics, Institute of Science and Technology*

Tribhuvan University, Kirtipur, Kathmandu, Nepal

Emails: subedi.abs@gmail.com, subedi_bh@cdmathtu.edu.np, bishnu.subedi@cdmathtu.edu.np, singh.ajaya1@gmail.com

Corresponding Author: Bishnu Hari Subedi

Abstract: *The subject of this paper is the well-known* $3n+1$ *problem of elementary number theory. This problem concerns with the behaviour of the iteration of a function which takes odd integers n to* $3n+1$, *and even integers n to* $\frac{n}{2}$. *There is a famous Collatz conjecture associated to this problem which asserts that, starting from any positive integer n, repeated iteration of the function eventually produces the value. We briefly discuss some basic facts and results of 3n + 1 problem and Collatz conjecture. Basically, we more concentrate on the generalization of this problem and conjecture to holomorphic dynamics.*

Keywords: 3*n* +1 *problem, Collatz conjecture, holomorphic dynamics, Fatou set, Julia set, (pre)-periodic Fatou component, wandering domain*.

AMS (2010) Subject Classification: 11Y55, 11B37, 30D05

1. Introduction

There are certain problems in mathematics which looks very simple and understandable to wide range of peoples, but very difficult to solve, and no one has yet found the solution to them. These problems are quite interesting because it seems the prerequisites for understanding the statement of the problem are much lower than the prerequisites for working on the problem. One of such problems is the $3n+1$ problem. This is a well-known problem in elementary number theory, and it can be explained to a child who has learned how to divide by 2 and multiply by 3. The problem can be stated simply as follows: Take any positive integer *n*. If *n* is even, divide it by 2 to get $\frac{n}{2}$. If *n* is odd, multiply it by 3 and add 1 to get $3n+1$. Repeat the process again and again. The conjecture associated to this problem is called *Collatz conjecture* or simply $3n+1$ *conjecture*, and it asserts that, no matter what the number *n* is taken, the process will always eventually reach 1, that is, every positive integer is eventually periodic, and the cycle it falls onto is $1 \rightarrow 4 \rightarrow 2 \rightarrow 1$.This conjecture is first proposed by Lothar Collatz in 1937. It is also known, the *Ulam conjecture*, the *Kakutani's problem*, the *Twaites conjecture*, or the *Syracuse problem*. The problem looks very simple, but "mathematics has not been ready for such problems", according to Paul Erdos **[6].** He also offered US$500 for its solution **[7].** Jeffrey Lagarias in 2010 claimed that, based only on known information about this problem, this is an extraordinarily difficult problem, completely out of research of present mathematics **[10].** Therefore, the Collatz conjecture remains today unsolved problem of mathematics, as it has been for over 80 years. The interest in this problem extends past the area of Number Theory; including Computer Science, via algorithms to help compute and find patterns in our iteration, into Logic as decision problems, and Dynamical Systems, by examining our iteration as a dynamical system on set of integer's $\mathbb{Z}$. A systematic survey with variety of already established results about $3n+1$ problem can be found in the work of Lagarias **[9], [10**], and Wirsching **[21].**The classical Collatz conjecture has been extensively studied by several researchers **[5], [16]**.

In this paper, we more concentrate on the fact of extending this problem to holomorphic dynamics. The related research had been done first by Letherman et al. **[13]** in 1999 and a little bit more extension by Lakshminarayanan and Ramohan **[11]** in 2012.

2. Mathematical Formulation of the 3*n* + 1 Problem

We first define some useful functions and concepts to describe behaviour of the sequences and starting values in the $3n + 1$ problem.

Definition 2.1 (3*n* + 1 function): Let $\mathbb{N}$ represents the set of natural numbers. For any $n \in \mathbb{N}$, the $3n + 1$ or Collatz function $f : \mathbb{N} \to \mathbb{N}$ is defined by $f(n) = 3n + 1$ if n is odd and $f(n) = \frac{n}{2}$ if n is even. In modular notation, function f can be written as $f(n) = 3n + 1$ if $n \equiv 1 \pmod 2$, and $f(n) = \frac{n}{2}$ if $n \equiv 0 \pmod 2$.

There are some terminologies which are defined by using $3n + 1$ function.

The *trajectory* or *orbit* $O^+(n)$ of n is the ordered set $\{n, f(n), f^2(n), f^3(n), \cdots\}$, where f^i represents i^{th} composition of f with itself, and it is usually known as i^{th} *iterates* of f. If $|O^+(n)| = \infty$, then $O^+(n)$ is said to be a *divergent trajectory*. If $|O^+(n)| = k < \infty$ and $f^k(n) = n$, then $O^+(n)$ is said to be a *cycle* of length k.

The number of steps needed to iterate below n: $\gamma(n) = \inf\{k : f^k(n) < n\}$ is called the *stopping time* of n. The number of steps needed to iterate 1: $\sigma(n) = \inf\{k : f^k(n) = 1\}$ is called the *total stopping time.* The largest number to which n iterates: $h(n) = \sup\{f^k(n) : k \in \mathbb{N}\}$ is called the *height* of n.

With these definitions, the $3n + 1$ or Collatz conjecture is formulated as follows.

Conjecture 2.1 (3*n* + 1 or Collatz conjecture): For every $n \in \mathbb{N}$, there exists a $k \in \mathbb{N}$ with $f^k(n) = 1$.

Conjecture 2.1 asserts that *every n has a finite total stopping time*. If, for some n, such a k doesn't exist, we say that n has infinite total stopping time and the conjecture is false. If the conjecture is false, it can only be because of there is some starting number which gives rise to a sequence that does not contain 1. Such a sequence would either enter a repeating cycle that excludes 1, or increase without bound. No such sequence has been found.

Let $a_k = f^k(n)$, then $a_0 = n$, $a_1 = f(n) = f(a_0)$, $a_2 = f(a_1) = f^2(n)$, and so on, with $a_0 \geq 1$. According to Conjecture 1.1, any $n \geq 1$ would always eventually arrive at $a_i = 1$ for some $i = 1, 2, \cdots$ after which it will stay in the cycle (1, 4, 2, 1) forever. For example, if we look at the starting values $a_0 = 1$ to 9, we get the following sequences:

$a_0 = 1$; $\{a_0, a_1, a_3, \cdots\} = \{1, 4, 2, 1, \cdots\}$.
$a_0 = 2$; $\{a_0, a_1, a_3, \cdots\} = \{2, 1, \cdots\}$.
$a_0 = 3$; $\{a_0, a_1, a_3, \cdots\} = \{3, 10, 5, 16, 8, 4, 2, 1, \cdots\}$.
$a_0 = 4$; $\{a_0, a_1, a_3, \cdots\} = \{4, 2, 1, \cdots\}$.
$a_0 = 5$; $\{a_0, a_1, a_3, \cdots\} = \{5, 16, 8, 4, 2, 1, \cdots\}$.
$a_0 = 6$; $\{a_0, a_1, a_3, \cdots\} = \{6, 3, 10, 5, 16, 8, 4, 2, 1, \cdots\}$.
$a_0 = 7$; $\{a_0, a_1, a_3, \cdots\} = \{7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, \cdots\}$.
$a_0 = 8$; $\{a_0, a_1, a_3, \cdots\} = \{8, 4, 2, 1 \ldots\}$.
$a_0 = 9$; $\{a_0, a_1, a_3, \cdots\} = \{9, 28, 14, 7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, \cdots\}$.

However, when we look at $a_0 = 27$, we get the following sequence: $\{a_0, a_1, a_2, \cdots\} = \{27, 82, 41, 124, 62, 31,$ $94, 47, 142, 71, 214, 107, 322, 161, 484, 242, 121, 364, 182, 91, 274, 137, 414, 206, 103, 310, 155, 466, 233,$

700, 350, 175, 526, 263, ⋯}. Surprisingly, it takes 111 iterations to reach 1, and the largest number that we get in 77^{th} iterations is 9232. That is, σ (27) = 111, h (27) = 9232, and γ (27) =96. As we see in the above sequence, the number 9 has longest total stopping time 19 for all $n \in \mathbb{N}$ below 10. Likewise, below 100, number 97 has longest total stopping time 118; below 1000, number 871 has longest total stopping time 178; below 10000, number 6171 has longest total stopping time 261. Below 10^{10}, number 9780657630 has longest total stopping time 1132**[12],** and for the numberless than 10^{17}, 93571393692802302 has longest total stopping time 2091**[17].** Until 2020, the conjecture has been checked by computer for all starting values upto $2^{68} \approx 2.95\times10^{20}$. Please note that the computer evidence is not a proof that the conjecture is true. As shown in the cases of the Pólya conjecture **[15]**, the Mertens conjecture, and Skewes' number (**[18], [19])**, counterexamples were found when using very large numbers.

Mathematically speaking, a (discrete) dynamical system is a state space X, together with a shift map f from X to itself. The iterates $f, f^2, f^3, \cdots$describe the dynamics of the system. In the 3n +1 dynamical system, the state space is the set of natural numbers $\mathbb{N} = \{1, 2, 3, \cdots\}$ and the shift map is the 3n + 1 map f. The 3n +1 conjecture 2.1 highlights the basic fact that even very simple equations can lead to amazingly complicated dynamics. In this paper, we behave 3n +1 problem as a discrete integer dynamics, and we see the latest results that are extended upto holomorphic dynamics. Lagarias **[9]** extended this problem to the set of rational numbers $\mathbb{Q}$ by defining the maps:

$$f_k(x) = \frac{(3x+k)}{2} \text{ if } x \equiv 1 \pmod 2, \text{ and}$$

$$f_k(x) = \frac{x}{2} \text{ if } x \equiv 0 \pmod 2, \text{ where } k \equiv \pm 1 \pmod 6, \text{ and } (x, k) = 1.$$

Tempkin **[20]** extended the 3n +1 problem to the set of real numbers $\mathbb{R}$ by defining the function:

$$f(x) = \begin{cases} -(5n-2)x + n(10n-3), & \text{if } x \in [2n-1, 2n],\ n \in \mathbb{N} \\ (5n+4)x - n(10n+7), & \text{if } x \in [2n, 2n+1],\ n \in \mathbb{N} \end{cases} \tag{2.1}$$

He proved that on each interval $[n, n+1]$, $n \in \mathbb{N}$, f has periodic points of every possible period. Chamberland studied similar extension to (2.1) by defining the function:

$$f(x) = \frac{x}{2}\cos^2\frac{\pi x}{2} + 3x + \frac{1}{2}\sin^2\frac{\pi x}{2} = x + \frac{1}{4} - \frac{(2x+1)}{4}\cos(\pi x) \tag{2.2}$$

He showed that any cycle on $\mathbb{R}$ must be locally attractive. This map has two attracting cycles, namely (1, 2) and (1.192531907⋯, 2.138656335⋯). Equivalent to 3n +1 problem, he conjecture that these are only two attracting cycle of f on $\mathbb{R}^+$.

Letherman et al. **[13]** extended the map (2.2) to the set of complex numbers $\mathbb{C}$ by defining the function:

$$f(z) = \frac{z}{2} + \frac{1}{2}(1-\cos \pi z)\left(z + \frac{1}{2}\right) + \frac{1}{\pi}\left(\frac{1}{2} - \cos \pi z\right)\sin \pi z + h(z)\sin^2 \pi z \tag{2.3}$$

for any entire holomorphic function h. Note that first two terms of (2.3) match with (2.2), and this function agrees on all integers with the 3n + 1 function. The map (2.3) is thus known as the *holomorphic* 3n + 1 *function*. Conjecture 2.1 was reformulated by them in this context as follows.

Conjecture 2.2(The holomorphic 3n + 1 conjecture): Iterating the function f of (2.3) on any positive integer will land at the number 1 after finitely many steps.

3. Dynamics of a Holomorphic Function

In this section, we briefly review the notion of holomorphic dynamics. For more details, we refer **[4], [8], [14].**The main purpose of this section is to provide a general background for the dynamics of the holomorphic 3*n* +1 function (2.3) of Section 2. Note that our holomorphic 3*n* +1 function (2.3) is transcendental entire, so we mainly concern with results related to transcendental dynamics.

Let *f* be a holomorphic function. We can define orbit and cycle of *f* as defined in Section 2. Note that $f^n(z)$ is well defined for all z except for a countable set which consists of the pole of *f*, $f^2, f^3, \dots, f^{n-1}$ when *f* is meromorphic. The study of the iterative sequence $(f^n(z))$ for various initial state z is known as *holomorphic dynamics*. The major concern of holomorphic dynamics is to study the fate of these orbits in the sense that fate is predictable or not. That is, the goal of studying holomorphic dynamics is to describe the asymptotic or long term behaviour of the sequence$(f^n(z))$.The dynamics of a holomorphic map mostly concerns with a dichotomy of the complex plane $\mathbb{C}$ into disjoint subsets where the sequence $(f^n(z))$ as $n\to\infty$ shows a normal or chaotic behaviour.

A subset of $\mathbb{C}$ where $(f^n(\mathrm{z}))$ as $n\to\infty$ behaves normally (in the sense of Montel) is known as *Fatou set F*(*f*), and its complement is called *Julia set* J(*f*). A maximum domain of normality of the iterates of *f*, that is, a connected component of the Fatou set is known as a *stable domain* or *Fatou component*. Fatou set is open by definition, and so its complement Julia set is closed. A Fatou component is simply or multiply connected. The following assertions are about the simply connected Fatou components.

Theorem 3.1: Let *f* be a transcendental entire function bounded on a curve *Γ* which tends to ∞. Then every component of *F* (*f*) is simply connected.

Theorem 3.2: Every unbounded Fatou component of a transcendental entire function is simply connected.

Theorem 3.3: Let *f* be a transcendental entire function such that *F* (*f*) contains an unbounded component *D*. Then every components of *F* (*f*) is simply connected.

The dynamics of a holomorphic function, in large extent, is determined by the periodicity of a point. A point *z* is called *periodic* if $f^n(z)$ = *z* for some positive integer *n*. The smallest *n* is called its *period*. In particular, if *f* (*z*) = *z*, then z is called *fixed point* (or *equilibrium position*) of *f*. The point *z* is called *pre-periodic* **(**or *eventually periodic*) if $f^{k+n}(z) = f^k(z)$ for some *k*, *n* > 0 and *strictly pre-periodic* if it is pre-periodic but not periodic. Let *z* is a periodic point of period *n* with $(f^n)'(\mathrm{z}) = \lambda$, where the prime (′) denotes the complex differentiation. The complex number λ obtained in this way is called *multiplier* or *eigenvalue*. The point *z* is called *attractive* or *stable* (or *super-attractive*) if $|\lambda| < 1$ (or $|\lambda| = 0$), and in this case, nearby points are attracted to the orbit under iteration by ***f*,** *repelling* or *unstable* if $|\lambda| > 1$ in which case, points close to the orbit move away, and indifferent (or *neutral***)** if $|\lambda| = 1 = e^{2\pi i\theta}$ in which case, iteration of nearby points stay near *z* but not converge to z. When θ is rational number (in this case $\lambda^n = 1$ for some integer *n*), the periodic point is called *parabolic* (or *rationally indifferent*), and the nearby dynamics are completely known. When θ is irrational (in this case $\lambda^n \neq 1$), the periodic point *z* is called *irrationally indifferent,* and there are certain values of θ, where nearby dynamics is still not known. It is known that a non-constant and non-linear entire function has at least two periodic points of period 1 or 2. This fact is generalized to the following assertion.

Theorem 3.4[3]: A (transcendental) entire function has infinitely many (repelling) periodic points of period *n* (or f^n has infinitely many fixed points) for all $n \geq 2$.

This assertion was first proved by Rosenbloom in 1948, and later it was corrected by adding 'repelling' before periodic by Bergweiler in 1991. From Theorem 3.1, one can say that transcendental entire function need not have fixed points. For example, $f(z) = e^z + z$ has no fixed points. Also, transcendental entire function need have attracting periodic points. For example, $f(z) = e^z$ has no attracting periodic points.

The periodic points that we categorized above are contained in Fatou or Julia sets as shown in the following assertion.

Theorem 3.5: Let f be a holomorphic function. Then $F(f)$ contains all (super) attracting periodic points and cycles, J (f) contains all repelling periodic points and cycles, all rationally indifference periodic points and cycles.

There are two types of points for which the inverse of holomorphic functions are not well defined, namely critical values and asymptotic values, and collectively they are known as singular values.

Definition 3.1(*Critical value, asymptotic value and singular value*): Let f be a holomorphic function. The *critical point* of a function f is a point z_0 such that $f'(z_0) = 0$ and `the *critical value* is the image of critical point under f. A point $w \in \mathbb{C}$ is said to be an *asymptotic value* if there is a curve γ tending to ∞ such that, along γ the values of $f(z)$ converges to w. The closure of the set of critical and asymptotic values is known as the set of *singular values.* This set is usually denoted by SV (f).

Note that among the entire functions, only transcendental entire functions may have asymptotic values. It is clear that polynomials cannot have finite asymptotic values. There are certain holomorphic functions whose finite asymptotic values can also be critical values. For example, $f(z) = z^2 e^{-z^2}$ has an asymptotic value 0, and critical values 0, $1/e$.

Definition 3.2(*Finite type and bounded type holomorphic function*): Let f be a holomorphic function. If SV (f) is finite, then f is said to be *finite type*. If SV (f) is bounded, then f is said to be *bounded type*.

Note that any finite type function is necessarily bounded type, but the converse may not hold. For example, $f(z) = z^2 e^{-z^2}$ is finite type transcendental entire function, and hence it is also bounded type. Later, in Section 4, we will discuss that the holomorphic $3n + 1$ function (2.3) is bounded type but not finite type.

There are dynamically important different Fatou components. For transcendental entire functions, a Fatou component U is one of the following type, for more details, we refer **[4], [8], [14]**.

- *Periodic component*: if $f^n(U) \subseteq U$. Minimum n is called the period of U.
- *Pre-periodic component*: if $f^n(U)$ is periodic for some integer n.
- *Wandering domain*: if $\{f^n(U)\}$ are disjoint for all n.

Periodic component U is one of the following types.

- *Immediate attracting basin*: if U contains (supper) attracting periodic point.
- *Parabolic* or *Leau domain*: if ∂U contains periodic point.
- *Siegel disk*: if U contains irrationally indifference periodic point.
- *Baker domain*: if U is unbounded and dynamics converge to ∞ locally uniformly.

There are certain Fatou components which are always simply connected as shown in the following assertions.

Theorem 3.6[1]: Let f be a transcendental entire map. Then every periodic or pre-periodic Fatou component is simply connected, and therefore any multiply connected Fatou component is bounded and wandering.

Theorem 3.7[2]: Let f be a bounded type transcendental entire function. Then all component of $F(f)$ are simply connected.

For certain holomorphic functions, Baker and wandering domains do not exists as shown in the following assertion.

Theorem 3.8 [2]: Finite type transcendental entire functions do not have wandering domains**,** and bounded type transcendental entire function do not have Baker domain.

For certain holomorphic function, Fatou set can be empty as shown in the following assertions.

Theorem 3.9: Let f be a finite type holomorphic function such that orbit of each point in SV (f) either is pre-periodic or converge to ∞. Then $F(f) = \emptyset$. In particular, if $f(z) = e^{z}$, then $F(f) = \emptyset$.

Definition 3.3(Forward, backward and completely invariant set**):** Let f be a function. A set E is said to be *forward invariant* if $f(E) \subseteq E$, *backward invariant* if $f^{-1}(E) \subseteq E$, and *completely invariant* if it is both forward and backward invariant.

For example, $f(z) = z^2$ has two completely invariant domains, namely $\{z \in \mathbb{C}: |z| < 1\}$ and $\{z \in \mathbb{C}: |z| > 1\}$ as Fatou components, and Julia set $J(f) = \{z \in \mathbb{C}: |z| = 1\}$ which is also completely invariant. This is an example of entire function which has a bounded and an unbounded completely invariant Fatou component.

However, in case of transcendental entire functions, we have the following assertions.

Theorem 3.10: Let f be a transcendental entire function**.** Then a completely invariant Fatou component U is unbounded, and $J(f) = \partial U$ and f has at most one completely invariant Fatou component.

For any holomorphic function, Fatou and Julia sets are themselves completely invariant as shown in the following assertion.

Theorem 3.11: Let f be a holomorphic map. Then Fatou and Julia sets are completely invariant.

4. Dynamics of the Holomorphic 3*n* +1 Function

In this section, we examine the dynamical behaviour of the holomorphic $3n + 1$ function (2.3) of Section 2.

We can easily check that $z = 0$ is a fixed point of the holomorphic $3n + 1$ function (2.3). If we differentiate holomorphic $3n + 1$ function (2.3), we get

$$f'(z) = [\frac{\pi}{2}(z + \frac{1}{2}) + 2\sin\pi z + 2\pi h(z)\cos\pi z + h'(z)\sin\pi z]\sin\pi z \tag{4.1}$$

From (4.1), we can say that all integers are critical points of function (2.3). Also, $|\lambda| = |f'(0)| = 0$. Therefore, $z = 0$ is a super attracting fixed point, and thus it is in the Fatou set $F(f)$ (by Theorem 3.5). If we consider function $h(z)$ vanishes from function $f(z)$, then function (4.1) reduces to

$$f'(z) = [\frac{\pi}{2}(z + \frac{1}{2}) + 2\sin\pi z]\sin\pi z. \tag{4.2}$$

For any $z = n + \delta$ with $|\delta| < \frac{1}{|2\pi^2 n|}$, $n \in \mathbb{Z}$, one can calculate $|f'(z)| < \frac{1}{2}$.

By these facts, Letherman et al. (**[13]**, Lemmas 3.1, 3.2) proved the following assertions.

Theorem 4.1 ([13], Lemmas 3.1 and 3.2): Let f be a holomorphic $3n + 1$ function (2.3). Then all integers are critical points. If entire function h in (2.3) vanishes everywhere, then all integers are in the Fatou Set.

From (4.1), we also can say that the holomorphic $3n + 1$ function (2.3) is not finite type, but of course, bounded type. We can use this fact to prove the following assertions.

Theorem 4.2([13], Proposition 3.4): Every Fatou component of the holomorphic $3n +1$ function (2.3) is simply connected.

Proof: The holomorphic $3n + 1$ function (2.3) is bounded type. Hence, by Theorem 3.7, every component of $F(f)$ is simply connected. □

Proposition 4.1: Holomorphic $3n +1$ function (2.3) has no Baker domains.

Proof: The function (2.3) is a bounded type transcendental entire function. By Theorem 3.8, it does not have Baker domain. □

Theorem 4.2 was stated by Letherman et al. **[13]** but its above short proof is ours. Their proof of this theorem is different and little bit long. We stated and proved Proposition 4.1 ourselves. We can also justify the essence of Proposition 4.1by the following assertion.

Theorem 4.3 (**[13]**, Proposition 3.6]): Let f be a holomorphic $3n +1$ function (2.3). No domain at infinity can intersect the real line. In particular, no integer can be in a domain at infinity.

Since orbit of any $n \in \mathbb{Z}$ under the holomorphic $3n + 1$ function (2.3) contained in $\mathbb{Z}$. And, since all inters are supper attracting because they are all critical points of f. No orbit in a Siegel disk is discrete, so there is no chance of consisting integers in the Siegel disk as in the following assertion.

Theorem 4.4(**[13]**, Lemma 3.3): Let f be a holomorphic $3n +1$ function (2.3). If a Fatou component of f corresponding to an attracting orbit contains an integer, then this orbit is super attracting. No Fatou component corresponding to rational indifference orbit or to a Siegel disk can contain integers.

The holomorphic $3n +1$ function (2.3) is not finite type, so it may or may not have wandering domains. By Theorem 4.1, all integers are in Fatou set, and by Theorem 3.5, (super) attracting periodic points and cycles are in Fatou set. Therefore, every integer is in the basin of attraction of a (super) attracting periodic orbit of integers, or in a wandering Fatou component. By Theorem 4.2, every Fatou component is simply connected, so there is no chance of existing multiply connected wandering domains. In this context, holomorphic $3n +1$ function can have simply connected wandering domains only when Conjecture 2.2 is disproved. Letherman et al. **[13]**. Conjecture 3.7] conjectured that wandering domain for such a function does not exist.

Conjecture 4.1: The holomorphic $3n +1$ function (2.3) has no simply connected wandering domain intersecting the integers.

If this conjecture was proved, then the holomorphic $3n + 1$ Conjecture 2.1 is proved, and then the $3n + 1$ Conjecture 2.1 is also proved. Finally, we can say that the fate of the famous Collatz conjecture depends upon the Conjecture 4.1.

References

[1] Baker I. N. (1975). The domain of normality of an entire function. *Ann. Acad. Sci. Fenn. Ser. A I Math*. **1:** 277-283.

[2] Baker I. N., Kotus J., & Lu Y. (1992). Iterates of meromorphic function IV: Critical finite functions, *Results in Math*, **22:** 651-656.

[3] Bergweiler W. (1991) Periodic points of entire functions: Proof of a conjecture of Baker. *Complex Variable*, **17:** 57-72.

[4] Bergweiler W. (1993). Iteration of meromorphic function. *Bull. Amer. Math. Soc*. **29** (2)**:** 131-188.

[5] Bourrington B., & Okano, A. (2019). New experimental investigation for the 3n +1 problem: The binary projection of the Collatz map. *Rose-Hulman Undergraduate Mathematics Journal*, **20 (1):** 1-32.

[6] Guy R. K. (2004). E17: *Permutation Sequences*: Unsolved problems in number theory (3rd Ed.). *Springer-Verlag*.

[7] Guy R. K. (1983). Don't try to solve these problems, *Amer. Math. Monthly*, **90 (1):** 35–41.

[8] Hua Y. & Yang C. C. (1998). *Dynamics of transcendental functions*. Gorden and Breach Science Publishers.

[9] Lagarias J. C. (1990). The set of rational cycles for the 3x + 1 problem. *Acta Arithmetica*, **56:** 33–53.

[10] Lagarias J. C. (2010). *The ultimate challenge: the* 3*x* + 1 *problem,* Providence, R.I.: American Mathematical Society. p.4.

[11] Lakshminarayanan E. S. & Ramohan R. (2012) A remark on the 3*n* + 1 problem and holomorphic dynamics.

[12] Leavens G. T. &Vermeulen M. (1992). 3*x* + 1 Search Programs, *Computers & Mathematics with Applications*, **24 (11):** 79–99.

[13] Letherman S., Schleicher D. & Wood R. (1999). The 3*n* + 1-Problem and holomorphic dynamics, *Experimental Mathematics*, **8(3):** 241–251.

[14] Morosawa S., Nishimura Y., Taniguchi M., & Ueda T. (1998),*Holomorphic Dynamics*, Cambridge University Press, Cambridge, UK.

[15] Pólya G. (1919),*Verschiedene Bemerkungen zur Zahlentheorie*, *Jahresbericht der Deutschen Mathematiker-Vereinigung (in German*), **28:** 31–40.

[16] Ren W. (2019). A new approach on proving Collatz conjecture, *Hindawi Journal of Mathematics* **4:** 1-12.

[17] Roosendaal E. (2020) 3x+1 *Delay Records*, Retrieved 14 March 2020.

[18] Skewes S. (1933), On the difference π (x) –li(x), *Journal of the London Mathematical Society*, **8:** 277–283.

[19] Skewes S. (1955), On the difference π (*x*) –li(*x*) (II), *Proceedings of the London Mathematical Society*, **5:** 48–70.

[20] Tempkin J. (1993). Some properties of continuous extensions of the Collatz function, *Draft of Circa*.

[21] Wirsching G. J. (1998). *The dynamical system generated by the* 3*n* + 1 *function*, Berlin, Springer (Lecture Notes in Mathematics series**:** 1681).

□□