On the Degree of Approximation of a Function by Nörlund Means of its Fourier Laguerre Series

Suresh Kumar Sahani\(^1\), Vishnu Narayan Mishra\(^2\) and Narayan Prasad Pahari\(^3\)

\(^1\)Department of Mathematics, MIT Campus and Rajarshi Janak Campus, T.U, Janakpur, Nepal
\(^2\)Department of Mathematics, Indira Gandhi National Tribal University, Lalpur, Amarkantak, Anuppur, Madhya Pradesh, India
\(^3\)School of Mathematical Sciences, Tribhuvan University, Kathmandu, Nepal

Email: sureshkumarsahani35@gmail.com\(^1\), vishnunarayann Mishra@gmail.com\(^2\), nppahari@gmail.com\(^3\)

Corresponding Author: Suresh Kumar Sahani

Abstract: In this paper, we have proved the degree of approximation of function belonging to \(L[0,\infty)\) by Nörlund Summability of Fourier-Laguerre series at the end point \(x = 0\). The purpose of this paper is to concentrate on the approximation relations of the function in \(L[0,\infty)\) by Nörlund Summability of Fourier-Laguerre series associated with the given function motivated by the works \([3]\), \([9]\) and \([13]\).

Keywords: Nörlund methods, Fourier series, Summability.

2010 Mathematics subject Classification: 40G05, 40D05, 42A24

1. Introduction

The concepts of product summability methods are more powerful than the individual summability methods and it gives an approximation for wider class of functions than the individual methods \([6]\). A bulk number of researchers have studied the degree of approximation of a function using different summability means of its Fourier-Laguerre series \([1] , [5] , [7] , [10] , [11] , [14] \) and \([15]\). This can be done at the point after replacing the continuity condition in Szegö theorem by much lighter conditions.

Let \(f(t)\) be a Lebesgue measurable function in the interval \((0, \infty)\) such that the integral

\[
\int_0^\infty e^{-x} x^\alpha f(x) L_n^{(\alpha)}(x) dx, \quad \alpha > -1
\]

exists, where \(L_n^{(\alpha)}(x)\) is the \(n^{th}\) Laguerre polynomial of order \(\alpha > -1\) defined the generating function

\[
\sum_{n=0}^\infty L_n^{(\alpha)}(x) \omega^n = (1 - \omega)^{-\alpha} e^{-x/\omega} \quad \omega \neq 0
\]

The Fourier series of the function \(f(x)\) is given by

\[
f(x) \sim \sum_{n=0}^\infty a_n L_n^{(\alpha)}(x), \quad \alpha > -1,
\]

where the coefficients \(a_n\) are the defined by the following formulae

\[
\Gamma(\alpha + 1) A_n^\alpha a_n = \int_0^\infty e^{-x} x^\alpha f(x) L_n^{(\alpha)}(x) dx
\]
where
\[ A_n^\alpha = \left( \frac{n + a}{n} \right) \sim n^\alpha. \]

Also, we write
\[ \phi(x) = \frac{1}{\Gamma(\alpha+1)} e^{-x} x^\alpha \] (1.5)

Let \( \sum u_n \) be a given infinite series with the sequence \( \{S_n\} \) of its partial sums. Let \( \{p_n\} \) be a sequence of real or complex constants with \( p_n \) as its non-vanishing \( n \)th partial sum. The sequence to sequence transformation is given by
\[ \tau_n = \frac{1}{p_n} \sum_{m=0}^{n} p_{n-m} S_m. \] (1.6)

which defines the sequence of Nörlund means of the sequence \( \{S_n\} \), generated by the sequence of coefficients \( \{p_n\} \). If \( \tau_n \to s \) then \( \sum u_n \) or the sequence \( \{S_n\} \) to the sum \( S \). If the sequence of Nörlund mean \( \{\tau_n\} \) is of bounded variation i.e. if
\[ |\tau_n - \tau_{n-1}| < \infty \] (1.7)

then the series \( \sum u_n \) is said to be absolute Nörlund summable [5], [6] and [12]. The absolute Nörlund summability of a Fourier series has been studied by several researchers like Bor [5], [7], Mazhar [10], Fadden [11], Padhy, Tripathi and Mishra [14] and Siddiqui, Gi, Bohar and Brono [15].

In 1976, Yadav [19] was the first to establish a result on absolute Nörlund summability of Laguerre series at origin and in 1979, Beohar and Jadiya [2] established a result on the absolute Nörlund summability of Laguerre series at the point \( x = 0 \). Later on, several researchers like Alghamdi & Mursaleen [1], Beohar and Jadiya [2], [3], Khatri and Mishra [8], Karasniqi [9], Nigam and Sharma [13], Shanker [16], Tiwari and Kachhara [18] obtained the degree of approximations of \( L[0, \infty) \) of the Fourier-Laguerre series by Cesaro mean Nörlund, Euler, (C,1)(E,q),(C,2)(E,q) harmonic-Euler mean et al. Concerning the absolute Nörlund summability of Laguerre series Yadav [19] has proved the following theorem:

**Theorem 1.1:** If \( \{p_n\} \) be a non-negative and non-increasing sequence of constant such that
\[ p_n = \sum_{r=0}^{n} p_r \to \infty \] and \( \sum_{n} \frac{(-1+2\alpha)}{\rho_n} \) is convergent. i.e. \( \sum_{n} \frac{(-1+2\alpha)}{\rho_n} \) is convergent, then the Fourier series (3) is \( |N, p_n| \) summable at the end point \( x = 0 \), provided that for some small positive \( \epsilon \),
\[ \phi(x) \epsilon BV[0, \infty) \] (1.8)

where
\[ \int_{R} e^{-\frac{x}{2}} x^{\frac{a}{2} - \frac{7}{12}} |f(x)| dx = 0 \left( \frac{n^{-1}}{n} \right) \] and \( F(x) \) is bounded in \([0,\epsilon]\).
(1.9)

Where,
\[ \Phi(x) = \left| f(x) - f(0) \right| e^{-x} x^\alpha \] (1.10)

and,
\[ F(x) = \left| f(x) - f(0) \right| e^{-x} x^{\frac{(-1+2\alpha)}{\Gamma(\alpha+1)}} \] (1.11)
2. Main Results

Before proceeding with the main result, we begin with recalling some Lemmas that are required to prove main theorem in this paper.

**Lemma 2.1:** (Szego [17], p.177) Let \( \alpha \) be arbitrary and real, \( c \) and \( w \) be fixed positive constants, and let \( n \to \infty \). Then

\[
L_n^{(\alpha)}(x) = \begin{cases} \frac{1}{n} \left( \frac{1}{2} \right), & \text{if } 0 \leq x \leq \frac{c}{n} \\ o(n), & \text{if } 0 \leq x \leq \frac{c}{n} \end{cases}
\]

(2.1)

**Lemma 2.2:** (Szego [17], p. 240) Let \( \alpha \) be arbitrary and real, \( w > 0 \), \( 0 < \eta < 4 \). Then we have for \( n \to \infty \),

\[
\max e^{\frac{-x}{w}}. x^{\frac{2\alpha+1}{4}}, |L_n^{(\alpha)}(x)| \sim \begin{cases} \frac{\alpha}{n^\frac{3}{2}}, & \text{if } w \leq x \leq (4-\eta)n \\ \frac{\alpha}{n^\frac{3}{2}}, & \text{if } x \geq w \end{cases}
\]

(2.2)

**Lemma 2.3:** (Bhatta [4]) Let \( \sum a_n \) be an infinite series with \( S_n \) as its \( n^{th} \) partial sum and \( \{p_n\} \) be a non-increasing sequence such that \( p_n \to \infty \). If \( \sum_n \frac{|a_n|}{p_n} < \infty \) i.e. convergent, then the series \( \sum_n a_n \) is \( [N,P_n] \) summable.

Considering the above facts, we prove the following theorem:

**Theorem 2.4:** Let \( \chi(x) \) be a non-negative, and non-increasing function of \( x \) such that \( x^n\chi\left(\frac{1}{x}\right) \to 0 \) as \( n \to 0 \). Let \( \{p_n\} \) be a non-negative and monotonic non-increasing sequence of constants with \( P_n \) as its non-vanishing \( n^{th} \) partial sum such that \( \sum_n \frac{|\chi(n)|}{p_n} < \infty \) i.e. convergent. Let \( \frac{n}{2} > \alpha > -1 \) and \( w \) is a fixed positive constant. If

\[
\int_x^w \frac{\phi(s)}{S^{\alpha+1}} \, ds = o\left[ \frac{1}{<x/>} \right], \text{ as } x \to 0,
\]

(2.4)

\[
\int_x^n \frac{e^{x}}{S^{\alpha+1}} \chi^{\left(\frac{1}{x}\right)} \, dx = o\left[ \frac{\alpha}{n^\left(\frac{3}{2}\right)} \chi^{\left(\frac{1}{x}\right)} \right]
\]

(2.5)

and

\[
\int_n^\infty \frac{e^{x}}{S^{\alpha+1}} \chi^{\left(\frac{1}{x}\right)} \, dx = o\left[ \frac{\alpha}{n^\left(\frac{3}{2}\right)} \chi^{\left(\frac{1}{x}\right)} \right], \text{ as } n \to \infty,
\]

(2.6)

then the Fourier-Laguerre series (1.3) is \( [N,P_n] \) summable at the point \( x = 0 \).

**Proof:**

First of all, under the hypothesis (2.4), we prove

\[
\int_0^x |\phi(s)| \, ds = o\left[ x^\alpha \chi\left(\frac{1}{x}\right) \right]
\]

(2.7)

For this, we have

\[
I(s) = \int_x^w \frac{\phi(s)}{S^{\alpha+1}} \, ds = o\left[ \chi\left(\frac{1}{x}\right) \right]
\]

Then

\[
|\phi(s)| = -I'(s) \cdot S^\alpha
\]

Therefore,

\[
\int_0^x |\phi(s)| \, ds = -\int_0^x S^\alpha \cdot I'(s) \, ds
\]

\[
= -[S^\alpha I(S)]_0^x + (\alpha + 1) \int_0^x S^\alpha \cdot I(s) \, ds
\]

67
Then the proof of the theorem is as follows:

\[ \sum_{k=0}^{n} a_k L_k^{(\alpha)}(0) = o\left(\frac{1}{n^{\alpha+1}}\right) \]

Therefore

\[ S_n(0) = \sum_{k=0}^{n} a_k L_k^{(\alpha)}(0) \]

\[ = \sum_{k=0}^{n} \frac{1}{\Gamma(\alpha+1)} \int_{0}^{\infty} e^{-x} \cdot x^\alpha f(x) \sum_{k=0}^{n} L_k^{(\alpha)}(x) dx \]

\[ = \frac{1}{\Gamma(\alpha+1)} \int_{0}^{\infty} e^{-x} \cdot x^\alpha f(x) L_n^{(\alpha+1)}(x) dx \]

Again, due to the orthogonality of Laguerre polynomials, we have

\[ S_n(0) - f(0) = \int_{0}^{\infty} \phi(x) L_n^{(\alpha+1)}(x) dx \]

\[ = \left(\int_{0}^{c} + \int_{c}^{w} + \int_{w}^{n} + \int_{n}^{\infty}\right) \phi(x) L_n^{(\alpha+1)}(x) dx \]

\[ = I_1 + I_2 + I_3 + I_4 \quad \text{(say)} \]  

(2.9)

Next we consider \( I_1 \),

\[ |I_1| = \int_{0}^{n} |\phi(x)| L_n^{(\alpha+1)}(x) dx \]

\[ = o(n^{\alpha+1}) \int_{0}^{n} |\phi(x)| dx \]

\[ = o(\chi(n)), \quad \text{as } n \to \infty. \]  

(2.10)

Next we consider \( I_2 \), using (2.1) and (2.4), we have

\[ |I_2| = o\left(\frac{n^{\alpha+1}}{n^{\frac{\alpha+1}{2}}}ight) \int_{c}^{w} |\phi(x)| x^{\frac{\alpha+1}{2}} \frac{1}{\Gamma(\alpha+1)} dx \]

\[ = o\left(\frac{n^{\alpha+1}}{n^{\frac{\alpha+1}{2}}}ight) \int_{c}^{w} |\phi(x)| x^{\frac{\alpha+1}{2}} \frac{1}{\Gamma(\alpha+1)} dx \]

\[ = o(1) \cdot o(\chi(n)) \]

\[ = 0(\chi(n)), \quad \text{as } n \to \infty. \]  

(2.11)

Next we consider \( I_3 \), and using (2.2), (2.3) and (2.5), we have

\[ |I_3| = \int_{w}^{n} |\phi(x)| \left| L_n^{(\alpha+1)}(x) \right| dx \]

\[ = \int_{w}^{n} |\phi(x)| e^{\frac{x}{2^{(2\alpha+3)}}} \phi\left(\frac{2^{(2\alpha+1)}}{4}\right) dx \]

\[ = o\left(\frac{n^{\frac{2\alpha+1}{4}}}{4}\right) \int_{w}^{n} e^{\frac{x}{2^{(2\alpha+3)}}} \phi(x) dx \]
\[ = o \left( n^{\frac{2\alpha+1}{4}} \right) o \left( n^{\frac{-(2\alpha+1)}{4}} \right) \]
\[ = o[\chi(n)], \text{ as } n \to \infty \]  
(2.12)

Again, considering \( I_4 \), using (2.2), (2.6) and (2.7), we have
\[
|I_4| = \int_0^{\infty} |\phi(x)| e^{\frac{x}{4}} x^{\frac{-(2\alpha+1)}{4}} o \left( n^{\frac{6\alpha+5}{12}} \right) dx
\]
\[ = o \left( n^{\frac{6\alpha+5}{12}} \right) \int_0^{\infty} e^{\frac{x}{4}} x^{\frac{-(2\alpha+1)}{4}} |\phi(x)| dx
\]
\[ = o \left( n^{\frac{6\alpha+5}{12}} \right) \int_0^{\infty} e^{\frac{x}{4}} x^{\frac{-(6\alpha+7)}{12}} x^{\frac{-1}{6}} |\phi(x)| dx
\]
\[ = o \left( n^{\frac{6\alpha+5}{12}} \right) o \left( n^{\frac{-1}{6}} \right) \int_0^{\infty} e^{\frac{x}{4}} x^{\frac{-(6\alpha+7)}{12}} \frac{1}{6} |\phi(x)| dx
\]
\[ = o \left( n^{\frac{2\alpha+1}{4}} \right) o \left( n^{\frac{-(2\alpha+1)}{4}} \right) \chi(n) \]
\[ = o[\chi(n)], \text{ as } n \to \infty \]  
(2.13)

Now combining (2.10), (2.11), (2.12) and (2.13), and using \( f(0) = 0 \), we get
\[
|S_n(0) - 0| = o[\chi(n)], \text{ as } n \to \infty \text{ and therefore } |S_n| = o[\chi(n)], \text{ as } n \to \infty.
\]

Therefore applying (2.7), we have
\[
\sum_n \frac{|S_n|}{p_n} = \sum_n \frac{\chi(n)}{p_n} < \infty \quad \text{i.e. convergent.}
\]

This completes the proof of the theorem.

**Conclusion**

In this paper, we have proved a theorem related to the degree of approximation of function belonging to \( L[0, \infty) \) by Nörlund Summability of Fourier-Laguerre series at the end point \( x = 0 \). This work establishes some of the results that characterize the approximation relations of the function in \( L[0, \infty) \) by Nörlund Summability of Fourier-Laguerre series. In fact, these results can be used for further study in many practical problems in science and engineering.

**References**


Suresh K. Sahani, Vishnu N. Mishra, Narayan P. Pahari / On The Degree of Approximation of a by...


