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Half-Cauchy Inverse NHE Distribution: Properties and Applications

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Abstract:*In this paper, we have generated a new continuous distribution with three parameters called Half-Cauchy inverse NHE distribution. The statistical properties and characteristics of the proposed distribution like the hazard rate function (HRF), the probability density function (PDF), and the cumulative distribution function (CDF), quantile function also the skewness and kurtosis are discussed. The parameters of the proposed distribution are estimated using the least-square estimation (LSE), Cramer-Von-Mises (CVM) and maximum likelihood estimation (MLE) methods. All the computations are performed in R programming software. To assess the application of the proposed distribution, areal lifetime data set is analyzed and performed the goodness-of-fit. It is found that the proposed distribution performed better as compared to some existing distributions.*

Keywords:*Cramer-Von-Mises, Least-square estimation,Inverse NHE distribution, Half-Cauchy distribution.*

Introduction

The exponential distribution (ED) plays a significant role in the modeling of survival and reliability data in applied statistics and probability theory. It has the memoryless property and is a particular case of the geometric and gamma distributions. In addition it can be applied for the study of the Poisson point processes. The ED has been widely used as a basis distribution during the past few decades to construct a more adaptable family of distributions. Various researchers have given the modifications and extensions of the ED, including Nadarajah and Kotz [20], who defined the beta exponential and generalized exponential by (Gupta &Kundu, [13]), the reliability estimation of the generalized inverted ED was developed by (Abouammoh &Alshingiti,[2]),an extension of the ED by (Nadarajah & Haghighi, [19]),Kumaraswamy exponential (Cordeiro& de Castro, [9]).A new exponential-type model having decreasing, increasing, constant, bathtub-shaped and upside-down bathtub failure rate function has been defined by (Lemonte, [15]).Ristic and Balakrishnan [23] has been developed Gamma EE.Mahdavi and Kundu [16] have created a brand-new approach for extending distributions by using the ED.

The Alpha power transformed extended exponential distribution has recently been developed by (Hassan et al., [14]). A new extension of the exponential distribution with some statistical features has been described by (Almarashi et al., [3]). The Type II half-logistic exponentiated exponential distribution has been introduced by (Abdulkabir & Ipinyomi, [1]).Chaudhary and Kumar [4] has defined the extension of ED called the half logistic exponential extension distribution. Another extension of ED was presented by (Chaudhary et al.,[7]) named the truncated Cauchy power– exponential distribution. Chaudhary and Sapkota [5] has also defined a novel three-parameter model having bathtub-shaped hazard rate curve called a modified NHE distribution.

By breaking down the curve on the origin such that it will only take into account non-negative values, we consider the Half-Cauchy distribution in this work, which is a specific case of the Cauchy distribution. Chaudhary and Kumar [6] has defined a new distribution having three parameters using Half -Cauchy family of distribution named Half-Cauchy modified exponential distribution. Shaw [25] has used the Half-Cauchy distribution having a heavy-tailed, as an alternative to model spreading distances, since it can forecast more recurrent long-distance spreading events. In addition, the Half-Cauchy distribution is also used by (Paradiset al.,[21]) to model ringing data on tits having two species in Ireland and Britain. The cumulative distribution function (CDF) of the Half-Cauchy distribution-following random variable X is given by

$$R(x;\theta) = \frac{2}{\pi}\tan^{-1}\left(\frac{x}{\theta}\right), \;\; x > 0, \theta > 0. \tag{1}$$

and the probability density function (PDF) corresponding to (1) is,

$$r(x;\theta) = \frac{2}{\pi}\left(\frac{\theta}{\theta^2 + x^2}\right), \;\; x > 0, \theta > 0. \tag{2}$$

Therefore we are interested to generate new distribution using Half-Cauchy family of distribution. The family of distribution defined by (Ristic & Balakrishnan, [23]) whose CDF can be obtained as

$$F(x) = 1 - \int_0^{-\ln[G(x)]} r(t)\, dt \;\;, \tag{3}$$

Here $G(x)$ is the CDF of any baseline distribution and $r(t)$ is the PDF of any distribution. The family of Half-Cauchy distribution whose CDF can be defined by using $r(t)$ as PDF of Half-Cauchy distribution defined in (2) as

$$\begin{aligned} F(x) &= 1 - \int_0^{-\ln[G(x)]} \frac{2}{\pi}\frac{\theta}{\theta^2 + t^2}\, dt \\ &= 1 - \frac{2}{\pi}\arctan\left\{-\frac{1}{\theta}\ln\left[G(x)\right]\right\}. \end{aligned} \tag{4}$$

The PDF corresponding to (4) can be expressed as

$$f(x) = \frac{2}{\pi}\frac{g(x)}{\theta G(x)}\left[1 + \left\{-\frac{1}{\theta}\log G(x)\right\}^2\right]^{-1} \tag{5}$$

The rest part of this article is organized as follows. We define the Half-Cauchy inverse NHE distribution and some of their statistical properties are also presented. The estimation of the parameters of the proposed distribution is carried out using the three widely used estimation technique namely Cramer-Von-Mises (CVM), maximum likelihood estimators (MLE) and least-square (LSE) methods. The application of the proposed model is presented. Finally some concluding explanations are entered.

The Half-Cauchy Inverse NHE(HC-INHE) distribution

In this paper, we have taken the inverse NHE distribution (Tahir et al., [26]) as a parent distribution. The inverse NHE distribution's CDF can be expressed as

$$G(x) = \exp\left\{1-\left(1+\frac{\lambda}{x}\right)^{\beta}\right\}; \lambda > 0, x > 0, \beta > 0. \tag{6}$$

The PDF corresponding to (6) can be written as

$$g(x) = \beta\lambda x^{-2}\left(1+\frac{\lambda}{x}\right)^{\beta-1}\exp\left\{1-\left(1+\frac{\lambda}{x}\right)^{\beta}\right\} \tag{7}$$

Substituting (6) and (7) in (4) and (5) We obtain the CDF of the Half-Cauchy inverse NHE distribution, which is denoted as

$$F(x) = 1-\frac{2}{\pi}\arctan\left[-\frac{1}{\theta}\left\{1-\left(1+\frac{\lambda}{x}\right)^{\beta}\right\}\right];\ x > 0, \beta, \lambda, \theta > 0. \tag{8}$$

And the PDF of Half-Cauchy inverse NHE can be expressed as

$$f(x) = \frac{2}{\pi}\frac{\beta\lambda}{\theta}x^{-2}\left(1+\frac{\lambda}{x}\right)^{\beta-1}\left[1+\left[-\frac{1}{\theta}\left\{1-\left(1+\frac{\lambda}{x}\right)^{\beta}\right\}\right]^{2}\right]^{-1};\ x > 0, \beta, \lambda, \theta > 0. \tag{9}$$

Survival function

The survival function of HC-INHE is

$$S(x) = \frac{2}{\pi}\arctan\left[-\frac{1}{\theta}\left\{1-\left(1+\frac{\lambda}{x}\right)^{\beta}\right\}\right];\ x > 0, \beta, \lambda, \theta > 0. \tag{10}$$

Hazard rate function

Hazard rate function of HC-INHE model with parameters (β, λ, θ) is

$$h(t) = \frac{f(t)}{1-F(t)};\ 0 < t < \infty$$

$$= \frac{\beta\lambda}{\theta}x^{-2}\left(1+\frac{\lambda}{x}\right)^{\beta-1}\left\{\arctan\left[-\frac{1}{\theta}\left\{1-\left(1+\frac{\lambda}{x}\right)^{\beta}\right\}\right]\right\}^{-1}\left[1+\left[-\frac{1}{\theta}\left\{1-\left(1+\frac{\lambda}{x}\right)^{\beta}\right\}\right]^{2}\right]^{-1};\ x > 0 \tag{11}$$

Reverse hazard function of HC-INHE

The reverse hazard function of HC-INHE is

$$h_{rev}(x) = \frac{f(x)}{F(x)}$$

$$= \frac{2}{\pi}\frac{\beta\lambda}{\theta}x^{-2}\left(1+\frac{\lambda}{x}\right)^{\beta-1}\left\{1-\frac{2}{\pi}\arctan\left[-\frac{1}{\theta}\left\{1-\left(1+\frac{\lambda}{x}\right)^{\beta}\right\}\right]\right\}^{-1}\left[1+\left[-\frac{1}{\theta}\left\{1-\left(1+\frac{\lambda}{x}\right)^{\beta}\right\}\right]^{2}\right]^{-1} \tag{12}$$

Figure 1 shows the various PDF and hazard rate function shapes for the HC-INHE(β,λ,θ) with varied parameter values.

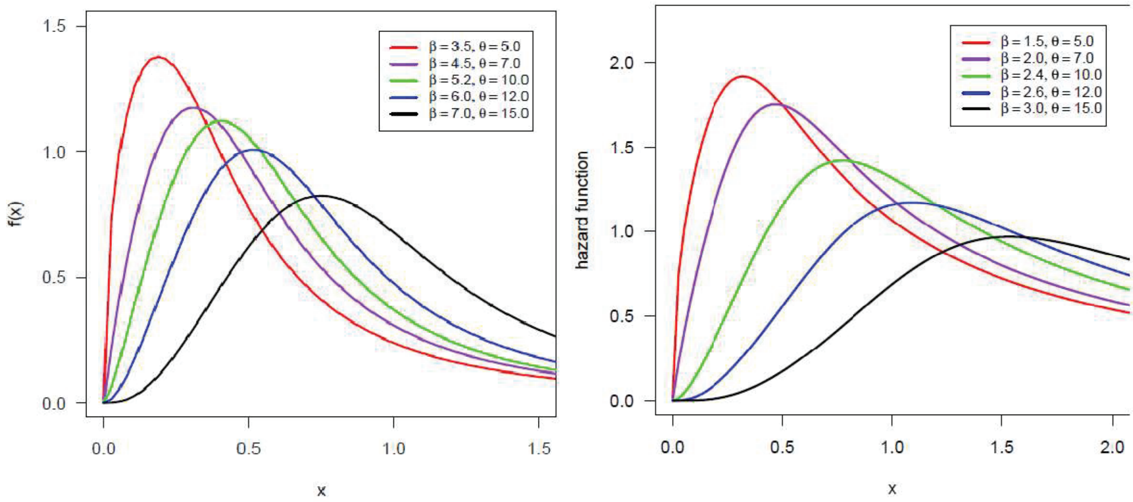

Figure 1. Graphs of the hazard function (right panel) and the PDF (left panel) for a fixed β and various values of λ and θ.

Cumulative hazard function (CHF)

The CHF of the HC-INHE(β,λ,θ) is defined as

$$H(x)=\int_{-\infty}^{x} h(y)dy$$

$$=-\log\left[1-F(x)\right]$$

$$=-\log\left[\frac{2}{\pi}\arctan\left[-\frac{1}{\theta}\left\{1-\left(1+\frac{\lambda}{x}\right)^{\beta}\right\}\right]\right];\ x>0 \qquad (13)$$

Quantile function

Let X be a non-negative random variable with CDF (10),then quantile function can be defined as

$$Q(u)=F^{-1}(u)$$

$$Q(u)=\lambda\left[\left\{1+\theta\tan\left(\frac{\pi(1-u)}{2}\right)\right\}^{1/\beta}-1\right]^{-1} \textit{ where } u\in(0,1) \qquad (14)$$

The HC-INHE(β,λ,θ)'s random deviate generation is,

$$x=\lambda\left[\left\{1+\theta\tan\left(\frac{\pi(1-v)}{2}\right)\right\}^{1/\beta}-1\right]^{-1} \textit{ where } \mathrm{v}\in(0,1) \qquad (15)$$

Median of HC-INHE can be calculated as

$$median=\lambda\left[\left\{1+\theta\right\}^{1/\beta}-1\right]^{-1}$$

Skewness and Kurtosis

The quartile-based Bowley's coefficient of skewness is

$$S_B=\frac{Q(3/4)-2Q(0.5)+Q(1/4)}{Q(3/4)-Q(1/4)}.$$

As stated by (Moors, [17]) using octiles, the coefficient of kurtosis is

$$K-Moors=\frac{Q(0.875)-Q(0.625)-Q(0.125)+Q(0.375)}{Q(3/4)-Q(1/4)}$$

Parameter estimation

Maximum Likelihood Estimation (MLE)

We have estimated the ML estimators (MLE's) of the HCINHE model by using MLE method. The log likelihood function can be expressed as, if $\underline{x}=(x_1,\ldots,x_n)$ is a random sample of size 'n' from $HCINHE(\beta,\lambda,\theta)$

$$\ell(\beta,\lambda,\theta\,|\,\underline{x})=n\ln(2/\pi)+n\ln\left(\frac{\beta\lambda}{\theta}\right)-2\sum_{i=1}^{n}\ln x_i+(\beta-1)\sum_{i=1}^{n}\ln A(x_i)-\sum_{i=}^{n}\ln\left\{1+\left[-\frac{1}{\theta}\left\{1-A(x_i)^{\beta}\right\}\right]^2\right\} \tag{16}$$

Differentiating (18) with respect to β, λ and θ, we get

$$\frac{\partial\ell}{\partial\beta}=\frac{n}{\beta}+\sum_{i=1}^{n}\ln A(x_i)\left\{1-\frac{2}{\theta}A(x_i)^{\beta}\left[-\frac{1}{\theta}\left\{1-A(x_i)^{\beta}\right\}\right]\left[1+\left[-\frac{1}{\theta}\left\{1-A(x_i)^{\beta}\right\}\right]^2\right]^{-1}\right\}$$

$$\frac{\partial\ell}{\partial\lambda}=\frac{n}{\lambda}+(\beta-1)\sum_{i=1}^{n}x_i^{-1}\left(A(x_i)\right)^{-1}-\frac{2\beta}{\theta}\sum_{i=1}^{n}x_iA(x_i)^{\beta-1}\left[-\frac{1}{\theta}\left\{1-A(x_i)^{\beta}\right\}\right]\left[1+\left[-\frac{1}{\theta}\left\{1-A(x_i)^{\beta}\right\}\right]^2\right]^{-1}$$

$$\frac{\partial\ell}{\partial\theta}=-\frac{n}{\theta}+\frac{2}{\theta}\sum_{i=1}^{n}\left[-\frac{1}{\theta}\left\{1-A(x_i)^{\beta}\right\}\right]^2\left[1+\left[-\frac{1}{\theta}\left\{1-A(x_i)^{\beta}\right\}\right]^2\right]^{-1}$$

where $A(x_i)=\left(1+\frac{\lambda}{x_i}\right)$.

Equating $\frac{\partial\ell}{\partial\beta}=\frac{\partial\ell}{\partial\lambda}=\frac{\partial\ell}{\partial\theta}=0$ and solving simultaneously for the β, λ and θ, the ML estimators of the $HCINHE(\beta,\lambda,\theta)$ model are obtained. But since solving the non-linear equations above is generally impossible, one can readily solve them with the aid of appropriate computer software.

Let $\underline{\Theta} = (\beta, \lambda, \theta)$ denote the parameter vector of $HCINHE(\beta, \lambda, \theta)$ and the corresponding MLE of $\underline{\Theta}$ as $\hat{\underline{\Theta}} = (\hat{\beta}, \hat{\lambda}, \hat{\theta})$ then the asymptotic normality results in, $\left(\hat{\underline{\Theta}} - \underline{\Theta}\right) \to N_3\left[0, \left(I(\underline{\Theta})\right)^{-1}\right]$ where $I(\underline{\Theta})$ is the Fisher's information matrix given by,

$$I(\underline{\Theta}) = -\begin{pmatrix} E\left(\dfrac{\partial^2 l}{\partial \beta^2}\right) & E\left(\dfrac{\partial^2 l}{\partial \beta \partial \lambda}\right) & E\left(\dfrac{\partial^2 l}{\partial \beta \partial \theta}\right) \\ E\left(\dfrac{\partial^2 l}{\partial \beta \partial \lambda}\right) & E\left(\dfrac{\partial^2 l}{\partial \lambda^2}\right) & E\left(\dfrac{\partial^2 l}{\partial \lambda \partial \theta}\right) \\ E\left(\dfrac{\partial^2 l}{\partial \beta \partial \theta}\right) & E\left(\dfrac{\partial^2 l}{\partial \lambda \partial \theta}\right) & E\left(\dfrac{\partial^2 l}{\partial \theta^2}\right) \end{pmatrix}$$

Since we are actually unsure of $\underline{\Theta}$, the asymptotic variance $\left(I(\underline{\Theta})\right)^{-1}$ of the MLE is useless. As a result, by entering the parameter estimates, we approximate the asymptotic variance. The information matrix $I(\underline{\Theta})$ given by the observed fisher information matrix $O(\hat{\underline{\Theta}})$ is used as

$$O(\hat{\underline{\Theta}}) = -\begin{pmatrix} \dfrac{\partial^2 l}{\partial \hat{\beta}^2} & \dfrac{\partial^2 l}{\partial \hat{\beta} \partial \hat{\lambda}} & \dfrac{\partial^2 l}{\partial \hat{\beta} \partial \hat{\theta}} \\ \dfrac{\partial^2 l}{\partial \hat{\beta} \partial \hat{\lambda}} & \dfrac{\partial^2 l}{\partial \hat{\lambda}^2} & \dfrac{\partial^2 l}{\partial \hat{\theta} \partial \hat{\lambda}} \\ \dfrac{\partial^2 l}{\partial \hat{\beta} \partial \hat{\theta}} & \dfrac{\partial^2 l}{\partial \hat{\theta} \partial \hat{\lambda}} & \dfrac{\partial^2 l}{\partial \hat{\theta}^2} \end{pmatrix}_{\left(\hat{\beta}, \hat{\lambda}, \hat{\theta}\right)} = -H(\underline{\Theta})_{\left(\underline{\Theta} = \hat{\underline{\Theta}}\right)}$$

where H is the Hessian matrix.

With the aim of maximizing likelihood, the Newton-Raphson method constructs the observed information matrix. The variance-covariance matrix is so displayed as follows:

$$\left[-H(\underline{\Theta})_{\left(\underline{\Theta} = \hat{\underline{\Theta}}\right)}\right]^{-1} = \begin{pmatrix} \operatorname{var}(\hat{\beta}) & \operatorname{cov}(\hat{\beta}, \hat{\lambda}) & \operatorname{cov}(\hat{\beta}, \hat{\theta}) \\ \operatorname{cov}(\hat{\beta}, \hat{\lambda}) & \operatorname{var}(\hat{\lambda}) & \operatorname{cov}(\hat{\lambda}, \hat{\theta}) \\ \operatorname{cov}(\hat{\beta}, \hat{\theta}) & \operatorname{cov}(\hat{\lambda}, \hat{\theta}) & \operatorname{var}(\hat{\theta}) \end{pmatrix} \tag{17}$$

Therefore, given that MLEs have asymptotic normality with the upper percentile of standard normal variate $Z_{\alpha/2}$, the 100(1- α) percent confidence intervals for estimating β, λ and θ of $HCINHE(\beta, \lambda, \theta)$ are formed as follows:

$$\hat{\beta} \pm Z_{\alpha/2}\sqrt{\operatorname{var}(\hat{\beta})},\ \hat{\lambda} \pm Z_{\alpha/2}\sqrt{\operatorname{var}(\hat{\lambda})}\ \text{ and } \hat{\theta} \pm Z_{\alpha/2}\sqrt{\operatorname{var}(\hat{\theta})}\ .$$

Method of Least-Square Estimation (LSE)

In order to estimate the unknown β, λ and θ parameters of the HCINHE distribution, we also used the least-square estimation method, which may be obtained by minimizing (18) with respect to unknown parameters β, λ and θ.

$$D(X; \beta, \lambda, \theta) = \sum_{i=1}^{n}\left[F(X_i) - \frac{i}{n+1}\right]^2 \tag{18}$$

Assume that $F(X_i)$ stands for the CDF of the ordered random variables $X_{(1)} < X_{(2)} < \ldots < X_{(n)}$, where $\{X_1, X_2, \ldots, X_n\}$ are n-samples drawn at random from a distribution function F. (.). By minimizing (21) with respect to β, λ and θ, the LSEs of β, λ and θ, say $\hat{\beta}, \hat{\lambda}$ *and* $\hat{\theta}$ are respectively found.

$$D(X;\beta,\lambda,\theta)=\sum_{i=1}^{n}\left[1-\frac{2}{\pi}\arctan\left[-\frac{1}{\theta}\left\{1-\left(1+\frac{\lambda}{x_i}\right)^{\beta}\right\}\right]-\frac{i}{n+1}\right]^2 \tag{19}$$

Differentiating (19) with respect to β, λ and θ we get,

$$\frac{\partial D}{\partial\beta}=\frac{2}{\theta}\sum_{i=1}^{n}\left[T(x_i)-\frac{i}{n+1}\right]\left(1+\frac{\lambda}{x_i}\right)^{\beta}\log\left(1+\frac{\lambda}{x_i}\right)\left[1+\{T(x_i)\}^2\right]^{-1}$$

$$\frac{\partial D}{\partial\lambda}=\frac{2\beta}{\theta}\sum_{i=1}^{n}\left[T(x_i)-\frac{i}{n+1}\right]\frac{1}{x_i}\left(1+\frac{\lambda}{x_i}\right)^{\beta-1}\left[1+\{T(x_i)\}^2\right]^{-1}$$

$$\frac{\partial D}{\partial\theta}=\frac{-2}{\theta^2}\sum_{i=1}^{n}\left[T(x_i)-\frac{i}{n+1}\right]\left\{1-\left(1+\frac{\lambda}{x_i}\right)^{\beta}\right\}\left[1+\{T(x_i)\}^2\right]^{-1}$$

where $T(x_i)=1-\frac{2}{\pi}\arctan\left[-\frac{1}{\theta}\left\{1-\left(1+\frac{\lambda}{x_i}\right)^{\beta}\right\}\right]$.

In a similar manner, the weighted least square estimators are calculated by minimizing (20) with respect to β, λ and θ.

$$B(X;\beta,\lambda,\theta)=\sum_{i=1}^{n}w_i\left[F(X_{(i)})-\frac{i}{n+1}\right] \tag{20}$$

Where $w_i = \text{weights} = \frac{1}{Var(X_{(i)})} = \frac{(n+1)^2(n+2)}{i(n-i+1)}$

Therefore, minimizing (21) with respect to β, λ and θ can be used to derive the weighted least square estimators of β, λ and θ, respectively.

$$B(X;\beta,\lambda,\theta)=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[1-\frac{2}{\pi}\arctan\left[-\frac{1}{\theta}\left\{1-\left(1+\frac{\lambda}{x_i}\right)^{\beta}\right\}\right]-\frac{i}{n+1}\right]^2 \tag{21}$$

Method of Cramer-Von-Mises estimation (CVME)

Through the process of minimizing the function (22), we have derived the Cramer-Von-Mises estimators of β, λ and θ.

$$\begin{aligned}C(X;\beta,\lambda,\theta)&=\frac{1}{12n}+\sum_{i=1}^{n}\left[F(x_{i:n}\mid\beta,\lambda,\theta)-\frac{2i-1}{2n}\right]^2\\&=\frac{1}{12n}+\sum_{i=1}^{n}\left[1-\frac{2}{\pi}\arctan\left[-\frac{1}{\theta}\left\{1-\left(1+\frac{\lambda}{x_i}\right)^{\beta}\right\}\right]-\frac{2i-1}{2n}\right]^2\end{aligned} \tag{22}$$

Differentiating (22) with respect to β, λ and θ we get,

$$\frac{\partial C}{\partial\beta}=\frac{2}{\theta}\sum_{i=1}^{n}\left[T(x_i)-\frac{2i-1}{2n}\right]\left(1+\frac{\lambda}{x_i}\right)^{\beta}\log\left(1+\frac{\lambda}{x_i}\right)\left[1+\{T(x_i)\}^2\right]^{-1}$$

$$\frac{\partial C}{\partial \lambda}=\frac{2\beta}{\theta}\sum_{i=1}^{n}\left[T(x_i)-\frac{2i-1}{2n}\right]\frac{1}{x_i}\left(1+\frac{\lambda}{x_i}\right)^{\beta-1}\left[1+\{T(x_i)\}^2\right]^{-1}$$

$$\frac{\partial C}{\partial \theta}=\frac{-2}{\theta^2}\sum_{i=1}^{n}\left[T(x_i)-\frac{2i-1}{2n}\right]\left\{1-\left(1+\frac{\lambda}{x_i}\right)^{\beta}\right\}\left[1+\{T(x_i)\}^2\right]^{-1}$$

where $T(x_i)=1-\frac{2}{\pi}\arctan\left[-\frac{1}{\theta}\left\{1-\left(1+\frac{\lambda}{x_i}\right)^{\beta}\right\}\right]$.

We arrive to the CVM estimators by simultaneously solving $\frac{\partial C}{\partial \beta}=0, \frac{\partial C}{\partial \lambda}=0$ *and* $\frac{\partial C}{\partial \theta}=0$.

Application to Real Dataset

Here, we used real data set to demonstrate how the HCINHE distribution is appropriate and applicable. The real data set, which details the duration of pain relief for 20 patients using analgesics, is taken from Gross and Clark [8] (p.105). Data are as follows:
1.4, 1.1, 1.7, 1.3, 1.8, 1.9, 2.2, 1.6, 2.7, 1.7, 1.8, 4.1, 1.2, 1.5, 3, 1.4, 2.3, 1.7, 2.0, 1.6

We estimated the MLEs of the HCINHE distribution by maximizing the likelihood function (16) using the R software's (R Core Team, [22]) optim() function (Dalgaard, [10]). Log-Likelihood value is l = -15.7949, which we have determined. In Table 1, the 95 percent asymptotic confidence interval (ACI) and the MLEs with their standard errors (SE) for β, λ and θ are presented.

Table 1:MLE and SE for β, λ and θ and 95%ACI

Parameter	MLE	SE	95%ACI
beta	79.7799	7.6345	(64.8163, 94.7435)
lambda	0.1129	0.0137	(0.0860, 0.1398)
theta	154.1769	9.3038	(135.9415, 172.4123)

As can be seen in the profile log-likelihood function graph for β, λ and θ in Figure 2, we have discovered that the ML estimates may be produced uniquely.

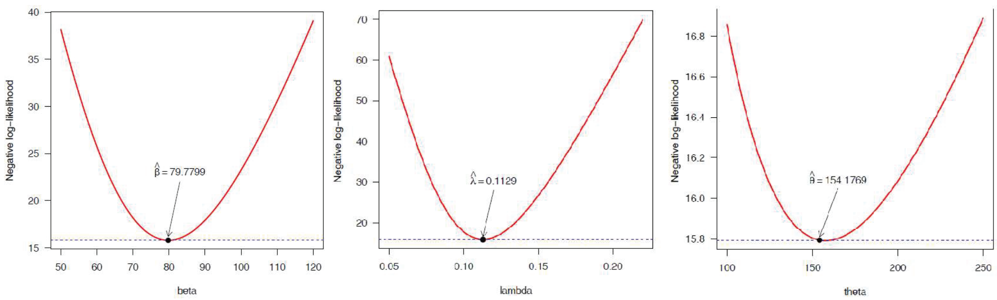

Figure 2: Log-likelihood function profile for the β, λ and θ.

The HCINHE distribution exhibits a good fit to the data, as evidenced by the P-P plot and Q-Q plot graphs in Figure 3.

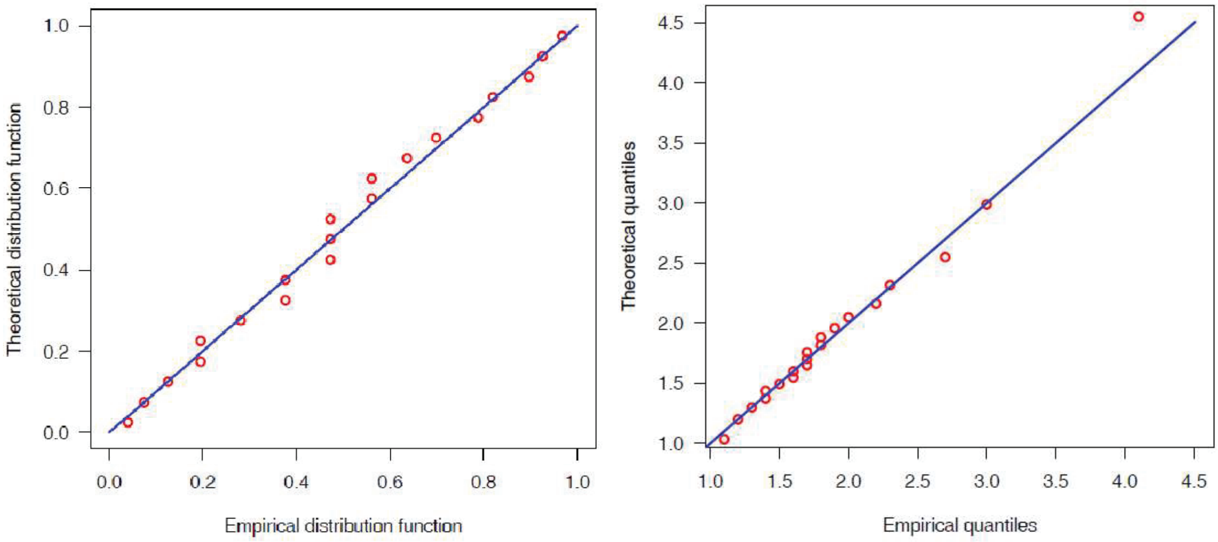

Figure 3: The HCINHE distribution's Q-Q plot (right panel) and P-P plot (left panel).

We have shown in Table 2 the estimated parameters values for HCINHE model, along with their related negative log-likelihoods and AIC criteria. We have used the MLE, LSE, and CVE methods to arrive at these results.

Table 2:AIC, log-likelihood, and estimated parameters

Method of Estimation	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$	LL	AIC
MLE	79.7799	0.1129	154.1769	-15.7949	37.5899
LSE	97.2431	0.0870	119.4002	-15.8864	37.7728
CVE	148.6391	0.0615	183.4452	-15.7362	37.4724

Table 3 displays the KS, W, and A^2 statistic for the MLE, LSE, and CVE estimators along with the corresponding p-values.

Table 3: The p-value for the KS, W, and A^2statistical analysis

Method of Estimation	*KS(p-value)*	*W(p-value)*	*A^2(p-value)*
MLE	0.0891(0.9974)	0.0206(0.9972)	0.1263(0.9997)
LSE	0.0876(0.9979)	0.0225(0.9951)	0.1414(0.9992)
CVE	0.0792(0.9996)	0.0193(0.9983)	0.1225(0.9998)

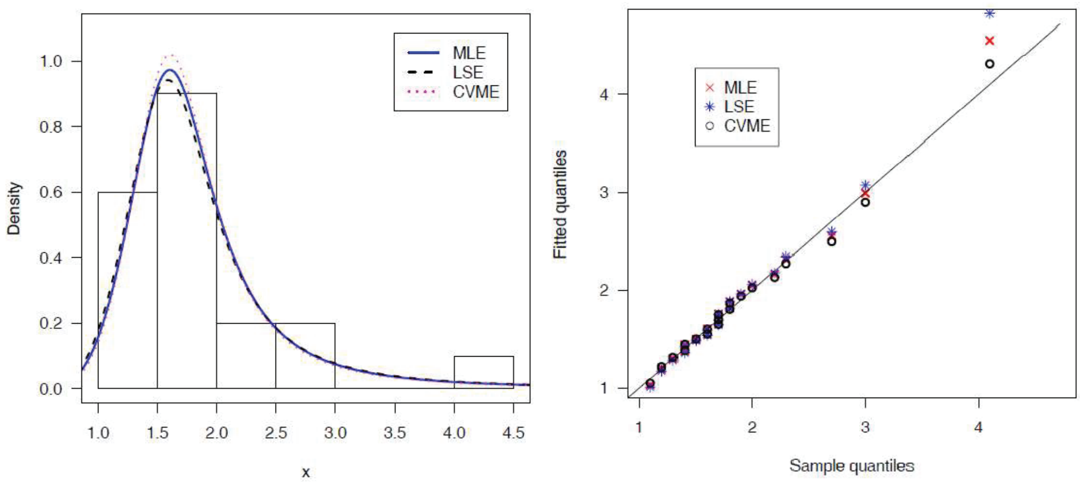

Figure 4: The Q-Q plot (right panel), Histogram, and density function of fitted distribution (left panel) of the HCINHE distribution using the MLE, LSE, and CVM estimation methods

Using a real dataset from past studies, we have demonstrated the applicability of the HCINHE distribution in this section. We have taken into account the five distributions namely Exponentiated Exponential Poisson (EEP) introduced by (Ristić & Nadarajah, [24]),Generalized Exponential Extension (GEE) distribution by (Lemonte, [15]), Weighted Lindley (WL) distribution by (Ghitany et al., [12]), Generalized Gompertz distribution by(El-Gohary et al., [11]) and Exponentiated Weibull Distribution (EW) by (Mudholkar & Srivastava, [18]) to compare the potentiality of the suggested model.

For the purpose of evaluating the usage of the HCINHE model, Table 4 provides instances of the Akaike information criterion (AIC), Bayesian information criterion (BIC), Corrected Akaike information criterion (CAIC), and Hannan-Quinn information criterion (HQIC).

Table 4

AIC, BIC, CAIC, HQIC, and Log-likelihood (LL)

Model	LL	AIC	BIC	CAIC	HQIC
HCINHE	-15.7949	37.5899	40.5772	39.0899	38.1731
EEP	-15.8259	37.6518	40.6390	39.1518	38.2349
EW	-15.8982	37.7963	40.7835	39.2963	38.3795
GEE	-16.0903	38.1807	41.1678	39.6807	38.7638
GGZ	-16.4717	38.9434	41.9306	40.4434	39.5266
WL	-17.8549	39.7099	41.7013	40.4158	40.0986

Figure 5 shows the chosen distributions and the graph of goodness-of-fit for the HCINHE distribution.

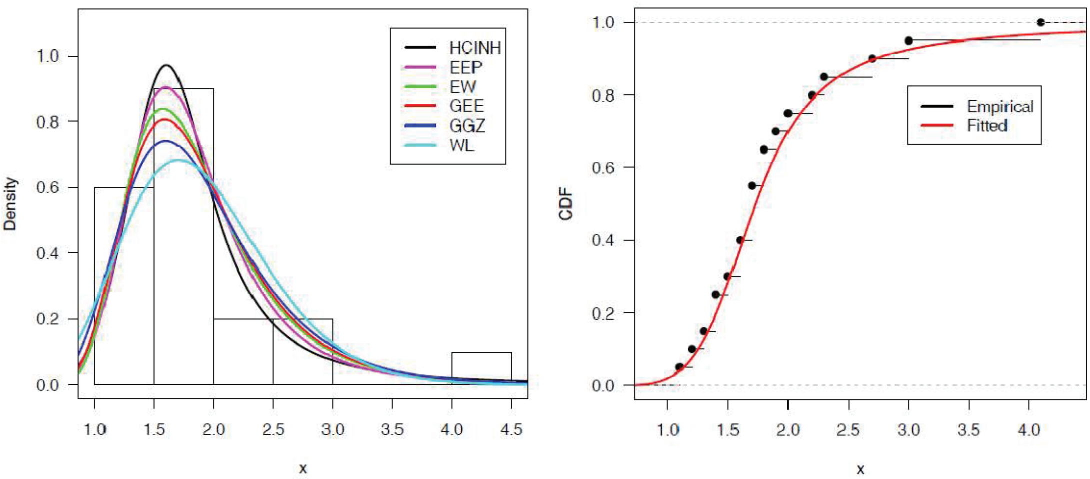

Figure 5: The HCINHE distribution's empirical distribution function with estimated distribution function (right panel) and histogram with the fitted distributions' density function (left panel).

We have also included the values of the Kolmogorov-Simnorov (KS), Anderson-Darling (AD), and Cramer-Von Mises (CVM) statistics in Table 5 to examine the goodness-of-fit of the HCINHE distribution in comparison to other competing distributions. As a result of the fact that the HCINHE distribution has a lower test statistic value and a higher p-value than other distributions used as comparisons, we can infer that it has a better fit to the data as well as results that are more consistent and reliable.

Table 5: Statistics for goodness-of-fit and the accompanying p-value

Model	*KS(p-value)*	*AD(p-value)*	*CVM(p-value)*
HCINHE	0.0891(0.9974)	0.0206(0.9972)	0.1263(0.9997)
EEP	0.1086(0.9723)	0.0299(0.9791)	0.2157(0.9856)
EW	0.1264(0.9065)	0.0412(0.9314)	0.2476(0.9715)
GEE	0.1350(0.8593)	0.0489(0.8882)	0.2850(0.9485)
GGZ	0.1446(0.7974)	0.0628(0.8020)	0.3564(0.8894)
WL	0.1741(0.5792)	0.1036(0.5720)	0.6055(0.6407)

Conclusion

A new distribution named Half-Cauchy inverse NHE distribution is presented in this article. The novel distribution's statistical properties, including the generation of accurate equations for its survival function, failure rate function, skewness, kurtosis, and quantile function are studied in detail. The parameter is estimated using three well-known estimation techniques:CVME, MLE and LSE. We discovered that MLEs perform relatively better than CVM and LSE techniques. The PDF of the suggested model's curves have demonstrated that it is versatile for modeling real-life data and may take on a variety of shapes, including increasing-decreasing and right-skewed. Additionally, the hazard function graph can be monotonically rising, reverse j-shaped or constant depending on the model parameters values. Using a real-life dataset to test the suggested model's applicability and adaptability, the results showed that it is considerably more flexible than certain other fitted distributions.

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