Abstract: Let $f$ be a map from $V(G)$ to \{0, 1, ..., $k$ – 1\} where $k$ is an integer, $1 \leq k \leq |V(G)|$. For each edge $uv$ assign the label $f(u)f(v) \text{(mod $k$)}$. $f$ is called a $k$-product cordial labeling if $|v_f(i) - v_f(j)| \leq 1$, and $|e_f(i) - e_f(j)| \leq 1$, $i, j \in \{0, 1, ..., k - 1\}$, where $v_f(x)$ and $e_f(x)$ denote the number of vertices and edges respectively labeled with $x$ ($x = 0, 1, ..., k - 1$). It is yet another study on $k$-product cordial labeling. In this paper, we define a new graph $P_n(t)$ namely Napier bridge graph and find some results on 3-product cordial and 4-product cordial labeling of Napier bridge graphs $P_n(3)$, $P_n(4)$ and $P_n(5)$.

Keywords: Cordial labeling, Product cordial labeling, $k$-Product cordial labeling, 3-Product cordial labeling, 4-Product cordial labeling, Napier bridge graph.

AMS Subject Classification (2010): 05C78.

1 Introduction

All graphs considered here are simple, finite, connected and undirected. We follow the basic notations and terminology of graph theory as in [3]. The concepts of labeling of graph has gained a lot of popularity in the field of graph theory during the last 60 years due to its wide range of applications. Labeling is a function that allocates the elements of a graph to real numbers, usually positive integers. In 1967, Rosa [15] published a pioneering paper on graph labeling problems. Thereafter, many types of graph labeling techniques have been studied by several authors. All these labelings are beautifully classified by Gallian [2] in his survey. Cordial labeling
is a weaker version of graceful and harmonious labeling was defined by Cahit [1]: Let $f$ be a function from the vertices of $G$ to $\{0, 1\}$ and for each edge $xy$ assign the label $|f(x) - f(y)|$. $f$ is called a cordial labeling of $G$ if the number of vertices labeled 0 and the number of vertices labeled 1 differ by at most 1, and the number of edges labeled 0 and the number of edges labeled 1 differ at most by 1. Motivated by the concept of cordial labeling, Sundaram et al. [16] introduced the concept of product cordial labeling: Let $f$ be a function from $V(G)$ to $\{0, 1\}$. For each edge $uv$, assign the label $f(u)f(v)$. Then $f$ is called product cordial labeling if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ where $v_f(i)$ and $e_f(i)$ denotes the number of vertices and edges respectively labeled with $i(i = 0, 1)$. Several results have been published on this topic (see [2]).

In 2012, Ponraj et al. [14] extended the concept of product cordial labeling and introduced k-product cordial labeling: Let $f$ be a map from $V(G)$ to $\{0, 1, ..., k - 1\}$ where $k$ is an integer, $1 \leq k \leq |V(G)|$. For each edge $uv$ assign the label $f(u)f(v)(\text{mod } k)$. $f$ is called a k-product cordial labeling if $|v_f(i) - v_f(j)| \leq 1$, and $|e_f(i) - e_f(j)| \leq 1$, $i, j \in \{0, 1, ..., k - 1\}$, where $v_f(x)$ and $e_f(x)$ denote the number of vertices and edges respectively labeled with $x (x = 0, 1, ..., k - 1)$. They proved that k-product cordial labeling of stars, bistars and also 4-product cordial labeling behavior of paths, cycles, complete graphs and combs. Javed and Jamil [4] proved that rhombic grid graphs are 3-total edge product cordial graphs. Jeyanthi and Maheswari [12] gave the maximum number of edges in a 3-product cordial graph of order $p$ is $\frac{p^2 - 3p + 8}{3}$ if $p \equiv 0(\text{mod } 3)$, $\frac{p^2 - 2p + 7}{3}$ if $p \equiv 1(\text{mod } 3)$ and $\frac{p^2 - p + 4}{3}$ if $p \equiv 2(\text{mod } 3)$ and also they showed that paths and cycles are 3-product cordial graphs. The same authors [13] proved that the graph $P_n^2$ is 3-product cordial. Inspired by the concept of k-product cordial labeling and also the results in [12, 13, 14], we further studied on k-product cordial labeling and established that the following graphs admit/ do not admit k-product cordial labeling: union of graphs [5]; cone and double cone graphs [6]; fan and double fan graphs [7]; powers of paths [8]; the maximum number of edges in a 4-product cordial graph of order $p$ is $4\left\lfloor \frac{p-1}{t-1} \right\rfloor \left\lceil \frac{p-1}{t-1} \right\rceil + 3$ [9]; product of graphs [10] and paths [11]. In this paper, we define a new graph $P_n(t)$ namely Napier bridge graph, since the image of the graph looks like the Napier bridge in Chennai city, India. This graph is obtained from the path $P_n$ by joining all the pairs of vertices $u, v$ of $P_n$ with $d(u, v) = t$. Clearly, $P_n(t) \cong P_n$ if $n \leq t$ and $P_n(t) \cong C_n$ if $n = t + 1$. In addition, we study the k-product cordial behavior of Napier bridge graph $P_n(t)$.

## 2 3-product cordial labeling of Napier bridge graphs

In this section, we study the 3-product cordial labeling of Napier bridge graphs $P_n(3)$, $P_n(4)$ and $P_n(5)$.
Theorem 2.1. For \( n \geq 5 \), the graph \( P_n(3) \) is 3-product cordial if and only if \( n \equiv 1 \) or \( 2 \) (mod 3).

Proof. Let the vertex and edge set of \( P_n(3) \) be \( V(P_n(3)) = \{v_i : 1 \leq i \leq n\} \) and 
\[
E(P_n(3)) = \{(v_i, v_{i+1}) : 1 \leq i \leq n - 1\} \cup \{(v_i, v_{i+3}) : 1 \leq i \leq n - 3\}
\]
respectively. We have the following three cases.

Define \( f : V(P_n(3)) \to \{0, 1, 2\} \) as follows:

Case (i): If \( n \equiv 1 \) (mod 3), then 
\[
f(v_i) = 0 \quad \text{for} \quad 1 \leq i \leq \lfloor \frac{n}{3} \rfloor.
\]
For \( i = \lfloor \frac{n}{3} \rfloor + j \), \( 1 \leq j \leq n - \lfloor \frac{n}{3} \rfloor \),
\[
f(v_i) = \begin{cases} 
1 & \text{if } j \equiv 1, 2 \text{ (mod 4)} \\
2 & \text{if } j \equiv 3, 0 \text{ (mod 4)}.
\end{cases}
\]

From the above labeling we get,
\[
v_f(0) + 1 = v_f(1) = v_f(2) + 1 = \lfloor \frac{n}{3} \rfloor + 1,
\]
\[
e_f(0) = e_f(1) + 1 = e_f(2) + 1 = 2\lfloor \frac{n}{3} \rfloor.
\]
Hence, \( P_n(3) \) is a 3-product cordial graph if \( n \equiv 1 \) (mod 3).

Case (ii): If \( n \equiv 2 \) (mod 3).

For \( n = 5 \),
\[
f(v_i) = \begin{cases} 
0 & \text{if } i = \lfloor \frac{n}{3} \rfloor \\
1 & \text{if } \lfloor \frac{n}{3} \rfloor + 1 \leq i \leq \lfloor \frac{n}{3} \rfloor + 2 \\
2 & \text{if } \lfloor \frac{n}{3} \rfloor + 3 \leq i \leq \lfloor \frac{n}{3} \rfloor + 4.
\end{cases}
\]

For \( n \geq 8 \),
\[
f(v_i) = \begin{cases} 
0 & \text{if } 1 \leq i \leq \lfloor \frac{n}{3} \rfloor \\
1 & \text{if } \lfloor \frac{n}{3} \rfloor + 1 \leq i \leq \lfloor \frac{n}{3} \rfloor + 3 \\
2 & \text{if } \lfloor \frac{n}{3} \rfloor + 4 \leq i \leq \lfloor \frac{n}{3} \rfloor + 6.
\end{cases}
\]

For \( i = \lfloor \frac{n}{3} \rfloor + j \), \( 1 \leq j \leq 2 \lfloor \frac{n}{3} \rfloor - 2 \),
\[
f(v_i) = \begin{cases} 
1 & \text{if } j \equiv 2, 4, 5, 7 \text{ (mod 8)} \\
2 & \text{if } j \equiv 1, 3, 6, 0 \text{ (mod 8)}.
\end{cases}
\]

Thus we get, 
\[
v_f(0) + 1 = v_f(1) = v_f(2) = \lfloor \frac{n}{3} \rfloor + 1,
\]
\[
e_f(0) = e_f(1) = e_f(2) = 2\lfloor \frac{n}{3} \rfloor.
\]
Hence, \( P_n(3) \) is a 3-product cordial graph if \( n \equiv 2 \) (mod 3) for \( n \geq 5 \).

Case (iii): If \( n \equiv 0 \) (mod 3) for \( n \geq 6 \), then \( |V(P_n(3))| = 3t \) and \( |E(P_n(3))| = 6t - 4 \).

Thus, \( v_f(i) = t \) (\( i = 0, 1, 2 \)) and \( e_f(i) = 2t - 1 \) or \( 2t - 2 \) (\( i = 0, 1, 2 \)). If \( v_f(0) = t \), 
then \( e_f(0) > 2t - 1 \) for \( t > 1 \). Therefore, \( |e_f(0) - e_f(j)| > 1 \) for \( j = 1, 2 \). Hence, \( P_n(3) \) 
is not a 3-product cordial graph if \( n \equiv 0 \) (mod 3) for \( n > 3 \). □
An example of 3-product cordial labeling of $P_f(3)$ is shown in Figure 1.

\[
\begin{array}{c}
\text{Figure 1: 3-product cordial labeling of } P_f(3).
\end{array}
\]

**Theorem 2.2.** For $n \geq 6$, the graph $P_n(4)$ is 3-product cordial if and only if $n \equiv 2 \pmod{3}$ or $n = 6$.

**Proof.** Let the vertex and edge set of $P_n(4)$ be $V(P_n(4)) = \{v_i : 1 \leq i \leq n\}$ and $E(P_n(4)) = \{(v_i, v_{i+1}) : 1 \leq i \leq n - 1\} \cup \{(v_i, v_{i+4}) : 1 \leq i \leq n - 4\}$ respectively. We have the following four cases.

**Case (i):** If $n \equiv 2(mod\ 3)$, then $f(v_i) = 0 : 1 \leq i \leq \lfloor \frac{n}{3} \rfloor$.

**Subcase (i):** If $n = 8$.

For $i = \lfloor \frac{n}{3} \rfloor + j ; 1 \leq j \leq n - \lfloor \frac{n}{3} \rfloor$,

\[
f(v_i) = \begin{cases} 
1 & \text{if } j \equiv 1, 0(mod\ 4) \\
2 & \text{if } j \equiv 2, 3(mod\ 4).
\end{cases}
\]

From the above labeling we get,

\[v_f(0) + 1 = v_f(1) = v_f(2) = \lfloor \frac{n}{3} \rfloor + 1,
\]

\[e_f(0) = e_f(1) = e_f(2) + 1 = 2\lfloor \frac{n}{3} \rfloor.
\]

**Subcase (ii):** If $n \geq 11$.

For $1 \leq i \leq 8$,

\[
f(v_i) = \begin{cases} 
1 & \text{if } i = 1, 2, 4, 5 \\
2 & \text{if } i = 3, 6, 7, 8.
\end{cases}
\]

For $i = \lfloor \frac{n}{3} \rfloor + 8 + j ; 1 \leq j \leq n - 8 - \lfloor \frac{n}{3} \rfloor$,

\[
f(v_i) = \begin{cases} 
1 & \text{if } j \equiv 1, 0(mod\ 4) \\
2 & \text{if } j \equiv 2, 3(mod\ 4).
\end{cases}
\]

From the above labeling we get,

\[v_f(0) + 1 = v_f(1) = v_f(2) = \lfloor \frac{n}{3} \rfloor + 1,
\]

\[e_f(0) = e_f(1) = e_f(2) + 1 = 2\lfloor \frac{n}{3} \rfloor.
\]

Hence, $P_n(4)$ is a 3-product cordial graph if $n \equiv 2(mod\ 3)$ for $n \geq 8$.

**Case (ii):** If $n = 6$, then the 3-product cordial labeling of $P_n(4)$ is shown in Table 1.
Table 1: 3-product cordial labeling of $P_n(4)$ for $n = 6$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$v_1$</th>
<th>$v_2$</th>
<th>$v_3$</th>
<th>$v_4$</th>
<th>$v_5$</th>
<th>$v_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

From the above labeling pattern we have, $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $i, j = 0, 1, 2$. Hence, $P_n(4)$ is a 3-product cordial graph if $n = 6$.

**Case (iii):** If $n \equiv 1 \text{(mod } 3) \text{ for } n \geq 7$, then $|V(P_n(4))| = 3t + 1$ and $|E(P_n(4))| = 6t - 3$. Thus, $v_f(i) = t$ or $t + 1$ ($i = 0, 1, 2$) and $e_f(i) = 2t - 1$ ($i = 0, 1, 2$). If $v_f(0) = t$ or $t + 1$, then $e_f(0) > 2t - 1$ for $t > 1$. Therefore, $|e_f(0) - e_f(j)| > 1$ for $j = 1, 2$. Hence, $P_n(4)$ is not a 3-product cordial graph if $n \equiv 1 \text{(mod } 3) \text{ for } n \geq 7$.

**Case (iv):** If $n \equiv 0 \text{(mod } 3) \text{ for } n \geq 9$, then $|V(P_n(4))| = 3t$ and $|E(P_n(4))| = 6t - 5$. Thus, $v_f(i) = t$ ($i = 0, 1, 2$) and $e_f(i) = 2t - 1$ or $2t - 2$ ($i = 0, 1, 2$). If $v_f(0) = t$, then $e_f(0) > 2t - 1$ for $t > 2$. Therefore, $|e_f(0) - e_f(j)| > 1$ for $j = 1, 2$. Hence, $P_n(4)$ is not a 3-product cordial graph if $n \equiv 0 \text{(mod } 3) \text{ for } n \geq 9$.

An example of 3-product cordial labeling of $P_5(4)$ is shown in Figure 2.

![Figure 2: 3-product cordial labeling of $P_5(4)$](image)

**Theorem 2.3.** For $n \geq 7$, the graph $P_n(5)$ is 3-product cordial if and only if $n \equiv 2 \text{ (mod } 3) \text{ or } n = 7$.

**Proof.** Let the vertex and edge set of $P_n(5)$ be $V(P_n(5)) = \{v_i : 1 \leq i \leq n\}$ and $E(P_n(5)) = \{(v_i, v_{i+1}) : 1 \leq i \leq n - 1\} \cup \{(v_i, v_{i+5}) : 1 \leq i \leq n - 5\}$ respectively. We have the following four cases.

**Define** $f: V(P_n(5)) \rightarrow \{0, 1, 2\}$ as follows:

**Case (i):** If $n = 7, 8$ or $11$, then the 3-product cordial labelings of $P_n(5)$ are shown in Table 2.

Table 2: 3-product cordial labelings of $P_n(5)$ for $n = 7, 8$ and $11$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$v_1$</th>
<th>$v_2$</th>
<th>$v_3$</th>
<th>$v_4$</th>
<th>$v_5$</th>
<th>$v_6$</th>
<th>$v_7$</th>
<th>$v_8$</th>
<th>$v_9$</th>
<th>$v_{10}$</th>
<th>$v_{11}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td></td>
<td></td>
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</tr>
<tr>
<td>11</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

From the above labeling pattern we have, $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $i, j = 0, 1, 2$. Hence, $P_n(5)$ is a 3-product cordial graph if $n = 7, 8$ or $11$. 

63
Case (ii): If $n \equiv 2 \pmod{3}$ for $n \geq 14$, then

$$f(v_i) = \begin{cases} 
0 & \text{if } 1 \leq i \leq \lfloor \frac{n}{3} \rfloor \\
1 & \text{if } \lfloor \frac{n}{3} \rfloor + 1 \leq i \leq \lfloor \frac{n}{3} \rfloor + 3 \\
2 & \text{if } \lfloor \frac{n}{3} \rfloor + 4 \leq i \leq \lfloor \frac{n}{3} \rfloor + 6.
\end{cases}$$

For $i = \lfloor \frac{n}{3} \rfloor + 6 + j ; 1 \leq j \leq n - 6 - \lfloor \frac{n}{3} \rfloor$,

$$f(v_i) = \begin{cases} 
1 & \text{if } j \equiv 2, 4, 5, 7 \pmod{8} \\
2 & \text{if } j \equiv 1, 3, 6, 0 \pmod{8}.
\end{cases}$$

From the above labeling we get,

$$v_f(0) + 1 = v_f(1) = v_f(2) = \lfloor \frac{n}{3} \rfloor + 1,$$

$$e_f(0) = e_f(1) + 1 = e_f(2) + 1 = 2\lfloor \frac{n}{3} \rfloor.$$ Hence, $P_n(5)$ is a 3-product cordial graph if $n \equiv 2 \pmod{3}$ for $n \geq 14$.

Case (iii): If $n \equiv 1 \pmod{3}$ for $n \geq 10$, then $|V(P_n(5))| = 3t + 1$ and $|E(P_n(5))| = 6t - 4$. Thus, $v_f(i) = t$ or $t + 1$ ($i = 0, 1, 2$) and $e_f(i) = 2t - 1$ or $2t - 2$ ($i = 0, 1, 2$). If $v_f(0) = t$ or $t + 1$, then $e_f(0) > 2t - 1$ for $t \geq 3$. Therefore, $|e_f(0) - e_f(j)| > 1$ for $j=1,2$. Hence, $P_n(5)$ is not a 3-product cordial graph if $n \equiv 1 \pmod{3}$ for $n \geq 10$.

Case (iv): If $n \equiv 0 \pmod{3}$ for $n \geq 9$, then $|V(P_n(5))| = 3t$ and $|E(P_n(5))| = 6t - 6$. Thus, $v_f(i) = t$ ($i = 0, 1, 2$) and $e_f(i) = 2t - 2$ ($i = 0, 1, 2$). If $v_f(0) = t$, then $e_f(0) > 2t - 2$ for $t \geq 3$. Therefore, $|e_f(0) - e_f(j)| > 1$ for $j=1,2$. Hence, $P_n(5)$ is not a 3-product cordial graph if $n \equiv 0 \pmod{3}$ for $n \geq 9$.

An example of 3-product cordial labeling of $P_8(5)$ is shown in Figure 3.

![Figure 3: 3-product cordial labeling of P_8(5).](image)

3 4-product cordial labeling of Napier bridge graphs

In this section, we study the 4-product cordial labeling of Napier bridge graphs $P_n(3)$, $P_n(4)$ and $P_n(5)$.

**Theorem 3.1.** For $n \geq 5$, the graph $P_n(3)$ is 4-product cordial if and only if $5 \leq n \leq 11$ except $n = 8$. 

64
From the above labeling pattern we have,

\[ P - t \]

Thus,

\[ i \]

not a 3-product cordial graph if

\[ n \]

Case (iii):

\[ \left\{ (v_i, v_{i+1}) ; 1 \leq i \leq n - 1 \right\} \cup \left\{ (v_i, v_{i+3}) ; 1 \leq i \leq n - 3 \right\} \]

respectively. We have the following five cases.

Define \( f : V(P_n(3)) \to \{0, 1, 2, 3\} \) as follows:

**Case (i):** If \( 5 \leq n \leq 11 \) except \( n = 8 \), then the 4-product cordial labelings of \( P_n(3) \) are shown in Table 3.

Table 3: 4-product cordial labelings of \( P_n(3) \) for \( 5 \leq n \leq 11 \) except \( n = 8 \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( v_0 )</th>
<th>( v_1 )</th>
<th>( v_2 )</th>
<th>( v_3 )</th>
<th>( v_4 )</th>
<th>( v_5 )</th>
<th>( v_6 )</th>
<th>( v_7 )</th>
<th>( v_8 )</th>
<th>( v_9 )</th>
<th>( v_{10} )</th>
<th>( v_{11} )</th>
</tr>
</thead>
<tbody>
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</tr>
</tbody>
</table>

From the above labeling pattern we have, \( |v_f(i) - v_f(j)| \leq 1 \) and \( |e_f(i) - e_f(j)| \leq 1 \) for all \( i, j = 0, 1, 2, 3 \). Hence, \( P_n(3) \) is a 4-product cordial graph if \( 5 \leq n \leq 11 \) except \( n = 8 \).

**Case (ii):** If \( n \equiv 0(\text{mod } 4) \) for \( n \geq 8 \), then \( |V(P_n(3))| = 4t \) and \( |E(P_n(3))| = 8t - 4 \). Thus, \( v_f(i) = t \ (i = 0, 1, 2, 3) \) and \( e_f(i) = 2t - 1 \ (i = 0, 1, 2, 3) \). If \( v_f(0) = t \), then \( e_f(0) > 2t - 1 \) for \( t \geq 2 \). Therefore \( |e_f(i) - e_f(j)| > 1 \) for all \( i, j = 0, 1, 2, 3 \). Hence, \( P_n(3) \) is not a 4-product cordial graph if \( n \equiv 0(\text{mod } 4) \).

**Case (iii):** If \( n \equiv 1(\text{mod } 4) \) for \( n \geq 13 \), then \( |V(P_n(3))| = 4t + 1 \) and \( |E(P_n(3))| = 8t - 2 \). Thus, \( v_f(i) = t \ or \ t + 1 \ (i = 0, 1, 2, 3) \) and \( e_f(i) = 2t \ or \ 2t - 1 \ (i = 0, 1, 2, 3) \). Clearly, \( v_f(0) = t \) and 0 must be assigned consecutively at the beginning or end of the path. Otherwise \( e_f(0) > 2t \). Thus, \( e_f(0) = 2t \). Now \( v_f(2) = t \ or \ t + 1 \). If \( v_f(2) = t \), then 2 must be assigned non-consecutively. Otherwise \( e_f(0) > 2t \). Then, \( e_f(2) > 2t \) for \( t \geq 3 \). Therefore \( |e_f(i) - e_f(j)| > 1 \) for all \( i, j = 0, 1, 2, 3 \). The similar argument shows that \( v_f(2) \) can not be \( t + 1 \). Hence, \( P_n(3) \) is not a 4-product cordial graph if \( n \equiv 1(\text{mod } 4) \) for \( n \geq 13 \).

**Case (iv):** If \( n \equiv 2(\text{mod } 4) \) for \( n \geq 14 \), then \( |V(P_n(3))| = 4t + 2 \) and \( |E(P_n(3))| = 8t \). Thus, \( v_f(i) = t \ or \ t + 1 \ (i = 0, 1, 2, 3) \) and \( e_f(i) = 2t \ (i = 0, 1, 2, 3) \). Clearly, \( v_f(0) = t \) and 0 must be assigned consecutively at the beginning or end of the path. Otherwise \( e_f(0) > 2t \). Thus, \( e_f(0) = 2t \). Now \( v_f(2) = t \ or \ t + 1 \). If \( v_f(2) = t \), then 2 must be assigned non-consecutively. Otherwise \( e_f(0) > 2t \). Thus, \( e_f(2) > 2t \) for \( t \geq 3 \). Therefore \( |e_f(i) - e_f(j)| > 1 \) for all \( i, j = 0, 1, 2, 3 \). The similar argument shows that \( v_f(2) \) can not be \( t + 1 \). Hence, \( P_n(3) \) is not a 4-product cordial graph if \( n \equiv 2(\text{mod } 4) \) for \( n \geq 14 \).
Case (v): If \( n \equiv 3(\text{mod } 4) \) for \( n \geq 15 \), then \( |V(P_n(3))| = 4t + 3 \) and \( |E(P_n(3))| = 8t + 2 \). Thus, \( v_f(i) = t \) or \( t + 1 \) \( (i = 0,1,2,3) \) and \( e_f(i) = 2t \) or \( 2t + 1 \) \( (i = 0,1,2,3) \). Obviously, \( v_f(0) = t \) and \( 0 \) must be assigned consecutively at the beginning or end of the path. Otherwise \( e_f(0) > 2t + 1 \). Thus, \( e_f(0) = 2t \). Clearly, \( v_f(2) = t + 1 \) and at most 2 consecutive vertices labeled with 2. Otherwise \( e_f(0) > 2t + 1 \). Then, \( e_f(2) > 2t + 1 \) for \( t \geq 3 \). Therefore \( |e_f(i) - e_f(j)| > 1 \) for all \( i,j = 0,1,2,3 \). Hence, \( P_n(3) \) is not a 4-product cordial graph if \( n \equiv 3(\text{mod } 4) \) for \( n \geq 15 \). 

An example of 4-product cordial labeling of \( P_5(3) \) is shown in Figure 4.

![Figure 4: 4-product cordial labeling of \( P_5(3) \).](image)

**Theorem 3.2.** For \( n \geq 6 \), the graph \( P_n(4) \) is 4-product cordial if and only if \( n = 6 \) or \( 10 \).

**Proof.** Let the vertex and edge set of \( P_n(4) \) be \( V(P_n(4)) = \{v_i : 1 \leq i \leq n\} \) and \( E(P_n(4)) = \{(v_i,v_{i+1}) : 1 \leq i \leq n-1\} \cup \{(v_i,v_{i+4}) : 1 \leq i \leq n-4\} \) respectively. We have the following five cases.

Define \( f : V(P_n(4)) \to \{0,1,2,3\} \) as follows:

Case (i): If \( n = 6 \) or \( 10 \), then the 4-product cordial labelings of \( P_n(4) \) are shown in Table 4.

<table>
<thead>
<tr>
<th>Table 4: 4-product cordial labelings of ( P_n(4) ) for ( n = 6 ) and ( 10 ).</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n )</td>
</tr>
<tr>
<td>---</td>
</tr>
<tr>
<td>6</td>
</tr>
<tr>
<td>10</td>
</tr>
</tbody>
</table>

From the above labeling pattern we have, \( |v_f(i) - v_f(j)| \leq 1 \) and \( |e_f(i) - e_f(j)| \leq 1 \) for all \( i,j = 0,1,2,3 \). Hence, \( P_n(4) \) is a 4-product cordial graph if \( n = 6 \) or \( 10 \).

Case (ii): If \( n \equiv 0(\text{mod } 4) \) for \( n \geq 8 \), then \( |V(P_n(4))| = 4t \) and \( |E(P_n(4))| = 8t - 5 \). Thus, \( v_f(i) = t \) \( (i = 0,1,2,3) \) and \( e_f(i) = 2t - 1 \) or \( 2t - 2 \) \( (i = 0,1,2,3) \). If \( v_f(0) = t \), then \( e_f(0) > 2t - 1 \) for \( t \geq 2 \). Therefore \( |e_f(i) - e_f(j)| > 1 \) for all \( i,j = 0,1,2,3 \). Hence, \( P_n(4) \) is not a 4-product cordial graph if \( n \equiv 0(\text{mod } 4) \) for \( n \geq 8 \).

Case (iii): If \( n \equiv 1(\text{mod } 4) \) for \( n \geq 9 \), then \( |V(P_n(4))| = 4t + 1 \) and \( |E(P_n(4))| = 8t - 3 \). Thus, \( v_f(i) = t \) or \( t + 1 \) \( (i = 0,1,2,3) \) and \( e_f(i) = 2t \) or \( 2t - 1 \) \( (i = 0,1,2,3) \). Clearly, \( v_f(0) = t \) and \( 0 \) must be assigned consecutively at the beginning or end of the path. Otherwise \( e_f(0) > 2t \). Thus, \( e_f(0) = 2t \). Now \( v_f(2) = t \) or \( t + 1 \). If
From the above labeling pattern we have, let the vertex and edge set of $P_n(4)$ be $V(P_n(4)) = \{v_i : 1 \leq i \leq n\}$ and $E(P_n(4)) = \{(v_i, v_{i+1}) : 1 \leq i \leq n-1\} \cup \{(v_i, v_{i+5}) : 1 \leq i \leq n-5\}$ respectively. We have the following five cases.

Define $f : V(P_n(5)) \rightarrow \{0, 1, 2, 3\}$ as follows:

**Case (i):** If $n = 7$ or 10, then the 4-product cordial labelings of $P_n(5)$ are shown in Table 5.

For $n \geq 7$, the graph $P_n(5)$ is 4-product cordial if and only if $n = 7$ or 10.

**Proof.** Let the vertex and edge set of $P_n(5)$ be $V(P_n(5)) = \{v_i : 1 \leq i \leq n\}$ and $E(P_n(5)) = \{(v_i, v_{i+1}) : 1 \leq i \leq n-1\} \cup \{(v_i, v_{i+5}) : 1 \leq i \leq n-5\}$ respectively. We have the following five cases.

Case (iv): If $n \equiv 2(\text{mod } 4)$ for $n \geq 14$, then $|V(P_n(4))| = 4t + 2$ and $|E(P_n(4))| = 8t - 1$. Thus, $v_f(i) = t$ or $t + 1$ ($i = 0, 1, 2, 3$) and $e_f(i) = 2t$ or $2t - 1$ ($i = 0, 1, 2, 3$). Clearly, $v_f(0) = t$ and 0 must be assigned consecutively at the beginning or end of the path. Otherwise $e_f(0) > 2t$. Thus, $e_f(0) = 2t$. Now $v_f(2) = t$ or $t + 1$. If $v_f(2) = t$, then 2 must be assigned non-consecutively. Otherwise $e_f(0) > 2t$. Then, $e_f(2) > 2t$ for $t \geq 3$. Therefore $|e_f(i) - e_f(j)| > 1$ for all $i, j = 0, 1, 2, 3$. The similar argument shows that $v_f(2)$ cannot be $t + 1$. Hence, $P_n(4)$ is not a 4-product cordial graph if $n \equiv 2(\text{mod } 4)$ for $n \geq 9$.

Case (v): If $n \equiv 3(\text{mod } 4)$ for $n \geq 7$, then $|V(P_n(4))| = 4t + 3$ and $|E(P_n(4))| = 8t + 1$. Thus, $v_f(i) = t$ or $t + 1$ ($i = 0, 1, 2, 3$) and $e_f(i) = 2t$ or $2t + 1$ ($i = 0, 1, 2, 3$). Clearly, $v_f(0) = t$ and 0 must be assigned consecutively at the beginning or end of the path. Otherwise $e_f(0) > 2t$. Thus, $e_f(0) = 2t$. Clearly, $v_f(2) = t + 1$ and at most 2 consecutive vertices labeled with 2. Otherwise $e_f(0) > 2t + 1$. Then, $e_f(2) \geq 2t + 1$ for $t \geq 1$. Therefore $|e_f(i) - e_f(j)| > 1$ for all $i, j = 0, 1, 2, 3$. Hence, $P_n(4)$ is not a 4-product cordial graph if $n \equiv 3(\text{mod } 4)$ for $n \geq 7$.

An example of 4-product cordial labeling of $P_6(4)$ is shown in Figure 5.

**Figure 5:** 4-product cordial labeling of $P_6(4)$.
Table 5: 4-product cordial labelings of $P_n(5)$ for $n = 7$ and 10.

<table>
<thead>
<tr>
<th>n</th>
<th>$v_1$</th>
<th>$v_2$</th>
<th>$v_3$</th>
<th>$v_4$</th>
<th>$v_5$</th>
<th>$v_7$</th>
<th>$v_8$</th>
<th>$v_{10}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>0</td>
<td>3</td>
<td>1</td>
</tr>
</tbody>
</table>

From the above labeling pattern we have, $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $i, j = 0, 1, 2, 3$. Hence, $P_n(5)$ is a 4-product cordial graph if $n = 7$ or 10.

**Case (ii):** If $n \equiv 0 \pmod{4}$ for $n \geq 8$, then $|V(P_n(5))| = 4t$ and $|E(P_n(5))| = 8t - 6$. Thus, $v_f(i) = t$ ($i = 0, 1, 2, 3$) and $e_f(i) = 2t - 1$ or $2t - 2$ ($i = 0, 1, 2, 3$). If $v_f(0) = t$, then $e_f(0) > 2t - 1$ for $t \geq 2$. Therefore $|e_f(i) - e_f(j)| > 1$ for all $i, j = 0, 1, 2, 3$. Hence, $P_n(5)$ is not a 4-product cordial graph if $n \equiv 0 \pmod{4}$ for $n \geq 8$.

**Case (iii):** If $n \equiv 1 \pmod{4}$ for $n \geq 9$, then $|V(P_n(5))| = 4t + 1$ and $|E(P_n(5))| = 8t - 4$. Thus, $v_f(i) = t + 1$ ($i = 0, 1, 2, 3$) and $e_f(i) = 2t - 1$ ($i = 0, 1, 2, 3$). If $v_f(0) = t + 1$, then $e_f(0) > 2t - 1$ for $t \geq 2$ Therefore $|e_f(i) - e_f(j)| > 1$ for all $i, j = 0, 1, 2, 3$. Hence, $P_n(5)$ is not a 4-product cordial graph if $n \equiv 1 \pmod{4}$ for $n \geq 9$.

**Case (iv):** If $n \equiv 2 \pmod{4}$ for $n \geq 14$, then $|V(P_n(5))| = 4t + 2$ and $|E(P_n(5))| = 8t - 2$. Thus, $v_f(i) = t + 1$ ($i = 0, 1, 2, 3$) and $e_f(i) = 2t$ or $2t - 1$ ($i = 0, 1, 2, 3$). Clearly, $v_f(0) = t$ and 0 must be assigned consecutively at the beginning or end of the path. Otherwise $e_f(0) > 2t$. Thus, $e_f(0) = 2t$. Now $v_f(2) = t + 1$. If $v_f(2) = t$, then 2 must be assigned non-consecutively. Otherwise $e_f(0) > 2t$. Then, $e_f(2) > 2t$ for $t \geq 3$. Therefore $|e_f(i) - e_f(j)| > 1$ for all $i, j = 0, 1, 2, 3$. The similar argument shows that $v_f(2)$ can not be $t + 1$. Hence, $P_n(5)$ is not a 4-product cordial graph if $n \equiv 2 \pmod{4}$ for $n \geq 14$.

**Case (v):** If $n \equiv 3 \pmod{4}$ for $n \geq 11$, then $|V(P_n(5))| = 4t + 3$ and $|E(P_n(5))| = 8t$. Thus, $v_f(i) = t + 1$ ($i = 0, 1, 2, 3$) and $e_f(i) = 2t$ ($i = 0, 1, 2, 3$). Clearly, $v_f(0) = t$ and 0 must be assigned consecutively at the beginning or end of the path. Otherwise $e_f(0) > 2t$. Thus, $e_f(0) = 2t$. Clearly, $v_f(2) = t + 1$ and 2 must be assigned non-consecutively. Otherwise $e_f(0) > 2t$. Then, $e_f(2) > 2t$ for $t \geq 2$. Therefore $|e_f(i) - e_f(j)| > 1$ for all $i, j = 0, 1, 2, 3$. Hence, $P_n(5)$ is not a 4-product cordial graph if $n \equiv 3 \pmod{4}$ for $n \geq 11$.

An example of 4-product cordial labeling of $P_7(5)$ is shown in Figure 6.
4 Conclusion

In this paper, we find the 3-product and 4-product cordial labeling of Napier bridge graphs $P_n(3)$, $P_n(4)$ and $P_n(5)$. In future, we propose to find the k-product cordial labeling of $P_n(m)$ for $k \geq 5$ and $m \geq 2$.

References


