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A Short Note in Quantum Continuity Equation

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Abstract: *In this article, we present a view that the interpretation of quantum mechanics lacks unanimous approval among physicists. To show its inadequacy, we describe the well-known measurement problem and discuss two solutions: an orthodox solution and hidden variables.*
We provide a brief overview of Bohmian mechanics. We then derive the continuity equation for quantum mechanics and show that a realist interpretation of the quantum probability current $\vec{J}$ *leads to the guiding equation for Bohmian mechanics.*

Keywords: Quantum continuity equation, Quantum probability current, Guiding equation.

1. Introduction

Before the advent of quantum mechanics, the basic understanding of natural phenomena came from classical mechanics; however, after the development of quantum mechanics, it superseded classical mechanics. We know quantum mechanics is more fundamental than classical mechanics, as classical mechanics emerges from quantum mechanics under appropriate limits. Historically, there were two independent formulations of quantum mechanics: *wave mechanics* (Schrödinger) and *matrix mechanics* (Heisenberg, Born, Jordan). Both founding fathers (Schrödinger and Heisenberg) thought that their version of the formulation was superior [7, 12].

Any scientific theory consists of two important parts: *formalism* and *interpretation.* Formalism deals with the mathematical aspect of the theory, which is essential in giving new predictions that can be tested. The interpretation provides the answers of the question of what exists at the fundamental level of reality [10]. The ontology of Newtonian mechanics is a particle. However, when one asks, "What is the ontology of quantum mechanics?" One cannot provide a unanimous answer agreed by most physicists and philosophers. Despite robust mathematical formalism, quantum mechanics has multiple interpretations that answer the question of "what exists?" It is a question whose answer is murky and clouded by various interpretations and the fancy structure they demand.

2. The Measurement Problem

Quantum mechanics postulates that the state vector in Hilbert space represents the state of a system that evolves from the Schrödinger equation, or in the light of matrix mechanics the evolution is described by a unitary operator. However, we get a conflict between what the dynamics predicts and what we observe; this is what we deem as "the measurement problem". We will present the problem following the approach given in [9] for a formal treatment [4].

Let's say we want to measure the electron spin. For this task, we have a sophisticated device that measures the spin and let *S* be the unitary operator associated with the physical observable spin.

If the electron is in $|\uparrow\rangle$ the device measures it as $|\uparrow\rangle$, similarly if it is in $|\downarrow\rangle$ then the device measures it as $|\downarrow\rangle$. Now instead of $|\uparrow\rangle$ or $|\downarrow\rangle$, we have $|\psi\rangle$ which is the superposition of both $|\uparrow\rangle$ and $|\downarrow\rangle$,

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle).$$

Now we apply the spin operator,

$$S|\psi\rangle = \frac{1}{\sqrt{2}}(S|\uparrow\rangle + S|\downarrow\rangle),$$

it then follows that the system should evolve in the following way:

$$\frac{1}{\sqrt{2}}(S|\uparrow\rangle + S|\downarrow\rangle) \Rightarrow \frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\rangle$$

which is a superposition of two spin states. But we would never observe a superposition instead, we observe the following:

$$either\ S|\uparrow\rangle = |\uparrow\rangle(\text{with frequency approximately } 1/2)$$

$$or$$

$$S|\downarrow\rangle = |\downarrow\rangle(\text{with frequency approximately } 1/2).$$

The predictions made by the dynamics and our observation contradict one another. What we get by measuring an electron that is in superposition is, sometimes we get $|\uparrow\rangle$ and sometimes $|\downarrow\rangle$ the best we can do is get the probability of finding the electron in spin $|\uparrow\rangle$ or $|\downarrow\rangle$, which is given by Born's rule.

3. Solutions

3.1 Orthodox Solution

One of the solutions to the measurement problem was given by Paul Dirac and John von Neumann, dubbed the orthodox solution [7, 8]. Neumann proposed that there are two kinds of processes happening in nature.

Process 1 is the process where the wave function collapses into a determinate value whose probability is given by the Born's rule; this is a stochastic, non-linear process.

Process 2. The wave faction evolves deterministically according to the Schrödinger equation.

Also, by introducing the concept of measurement into our laws, we are making this theory about observers, which does not sound right to invoke complex organisms as observers as part of its most basic framework [3].

3.2 Hidden Variables

A different interpretation of quantum mechanics, called Bohmian mechanics or hidden variables, was formulated by David Bohm in his seminal papers [5, 6]. Bohmian claims that the wave function does not give a complete description of reality [11].

The wave function is not fundamental and serves only as a pragmatic tool. Bohmian mechanics posits that the particle, which has a definite position, is more fundamental than the wave function. This particle is guided by the wave function.

Bohm's theory solves the measurement by positing determinate particle position, which does not involve the idea of measurement or observer as the fundamental part of the framework [3]. Now, we will present the postulate of Bohmian mechanics as presented in [1].

3.2.1 The state description

The description of a Bohmian state for a particle is specified by (ψ, q), where ψ is the wave function, and q is the actual position of the particle.

3.2.2 The dynamics

The Schrödinger equation gives the evolution of ψ:

$$i\hbar\frac{\partial\psi}{\partial t} = \widehat{H}\psi$$

where *i* is the complex number, h is reduced Planck constant, and $\widehat{H}$ is the Hamiltonian of the system. The evolution of the particle position is given by the guiding law (or equation of motion):

$$v = \frac{\nabla S}{m},$$

S is the phase of the wave function and *m* is the mass of the Bohmian particle.

3.2.3 The quantum equilibrium

The quantum equilibrium configuration probability distribution ρ for an ensemble of systems each having quantum state ψ is given by:

$$\rho = |\psi|^2. \tag{1}$$

4. Guiding Equation from Quantum Continuity Equation

The continuity equation in quantum mechanics is analogous to the continuity equation of electrodynamics and fluid mechanics. Let's take a single-particle system moving in one dimension in real potential *V*, then the Schrödinger equation is:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x,t)\psi, \tag{2}$$

and its complex conjugate is:

$$-i\hbar\frac{\partial\psi^*}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi^*}{\partial x^2} + V(x,t)\psi^*. \tag{3}$$

Multiplying equation (2) by ψ^*, multiplying equation (3) by ψ, and subtracting the second from the first we get

$$\frac{\partial|\psi|^2}{\partial t} = -\frac{\partial}{\partial x}\left[\frac{i\hbar}{2m}\left(\psi\frac{\partial\psi^*}{\partial x} - \psi^*\frac{\partial\psi}{\partial x}\right)\right]. \tag{4}$$

This takes the form of a continuity equation:

$$\frac{\partial\rho}{\partial t} = -\frac{\partial\vec{j}}{\partial x}. \tag{5}$$

We can easily generalize it for three dimensions:

$$\frac{\partial\rho}{\partial t} = -\nabla.\vec{j}, \tag{6}$$

Where $\rho \equiv |\psi|^2$is the probability density, and $\vec{j} \equiv \frac{i\hbar}{2m}(\psi\nabla\psi^* - \psi^*\nabla\psi)$ which is called *probability flux* or *quantum probability current.* In electrodynamics, the *continuity equation* represents the conservation of charge, which is not an independent assumption; rather, it is built into the laws of electrodynamics.

The continuity equation also provides a constraint in the values of ρ and $\vec{j}$, we cannot have any values of ρ and $\vec{j}$ they have to respect the conservation of charge [8].

Similarly, we can interpret equation (4) as representing local conservation of probability as $|\psi|^2$,which is probability density [4].

By using $z - z^* = 2i\, Im(z)$ where z is a complex number, and $Im(z)$ represents the imaginary part of z, then the quantum probability current $\vec{j}$ can be written as:

$$\vec{j} = -\frac{i\hbar}{2m}(\psi^*\nabla\psi - \psi\nabla\psi^*) = \frac{\hbar}{m}Im(\psi^*\nabla\psi). \tag{7}$$

Next, we decompose the wave function ψ into polar form:

$$\psi = Re^{\frac{i}{\hbar}S}. \tag{8}$$

where *R* and *S* are real-valued functions representing, respectively, the amplitude and phase of the wave function. We can express the probability density in terms of the amplitude as $\rho = R^2$ and make the following substitution:

$$\psi^*\nabla\psi = R\nabla R + \frac{i}{\hbar}R^2\nabla S. \tag{9}$$

The quantum probability current thus becomes:

$$\vec{j} = \frac{\hbar}{m} Im(R\nabla R + \frac{i}{\hbar} R^2 \nabla S) = \frac{\hbar}{m} \frac{R^2 \nabla S}{\hbar} = \frac{R^2 \nabla S}{m}, \quad (10)$$

which we can rewrite (using the relation $\rho = R^2$) as follows:

$$\vec{j} = \frac{\rho \nabla S}{m} = \rho v, \quad (11)$$

defining the velocity term as:

$$v \equiv \frac{\nabla S}{m}. \quad (12)$$

The implication of the quantum continuity equation with the realist interpretation of $\vec{j}$ gives us the guiding equation (11) of Bohmian mechanics. However, Sakurai and Napolitano warn the readers on the literal interpretation of $\vec{j}$ (as ρ times the velocity defined at every point in space) as a simultaneous measurement of position and velocity, which will violate the uncertainty principle [2].

5. Conclusion

We argued that quantum mechanics has a good and robust mathematical formalism and lacks a good interpretation of reality. We also presented a well-known problem in quantum mechanics: the measurement problem. And its possible solutions, in particular, we looked at the orthodox solution and why it does not give a satisfactory solution. We then explored another solution, the hidden variables, and how it solved the problem and presented the postulates of Bohmian mechanics. We then derived the quantum continuity equation and realistically interpreted the quantum probability current $\vec{j}$ and showed that by interpreting $\vec{j}$ as a real phenomenon, we arrive at the guiding equation in Bohmian mechanics.

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