Nepal Journal of Mathematical Sciences (NJMS)
ISSN: 2738-9928 (online), 2738-9812 (print)
Vol. 6, No. 1, 2025 (February): 45-50
DOI:10.3126/njmathsci.v6i1.77377

Research Article
Received Date: January 20, 2025
Accepted Date:March 26, 2025
Published Date: April 8, 2025

Extension of Hermite-Hadamard Type Integral Inequality for m-Convex Functions with Second Order Derivatives

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Abstract: *Integral inequality is a fascinating research domain that helps to estimate the integral mean of convex functions. The convexity theory plays a basic role in the development of various branches of applied sciences. Convexity and inequality are connected which has a fundamental character in many branches of pure and applied disciplines. The Hermite-Hadamard (H-H) type integral inequality is one of the most important inequalities associated with the convex functions. The researchers are being motivated to the extensions, enhancements and generalizations of H-H type inequality for different types of convex functions. In this paper, we have obtained an extension of some integral inequalities of Hermite-Hadamard type for m-convex functions with second order derivatives on the basis of the classical convex functions.*

Keywords: Convexity, m-convexity, integral inequality

1. Introduction

Convexity theory is essential in the theoretical aspects of mathematicians, economists, and physicists. Mathematicians utilize this theory to solve difficulties that emerge in several subjects of study. Convex analysis has played a pivotal role in the generalizations and extensions of inequalities theory over the last few decades. The theories of convexity and inequality are closely connected. Integral inequalities are important and valuable in information technology, integral operator theory, numerical integration, optimization theory, statistics, probability, and stochastic processes because they are elegant and effective. Many mathematicians and research academics have focused their efforts and contributions over the last few decades on studying these types of inequalities. Thus, for convexity, there is vast and important literature on inequalities. Convexity is a broad subject that also includes the theory of convex functions. Convexity is a powerful property of functions, also known as a natural property of functions. Furthermore,

its minimization property makes it unique, novel, and beneficial. Due to its minimization characteristic, it possesses a significant status in optimization theory, calculus of variation, and probability theory. So, the idea of convex functions has played a significant role in modern mathematics [2]. It has been noticed that several books and research articles have been published in the past few years. The H-H inequality substantially impacted the study of a convex function [1]. Numerous important inequalities have been employed as powerful tools not only in pure mathematics but also in other areas of mathematics, for if, the theory of means, approximation theory, numerical analysis, and so on. One of the most important inequalities that has been attracted by many inequality experts in the last few decades is the famous Hermite-Hadamard inequality. Although, it was firstly known in literature as a result of J. Hadamard in 1893, but this result was actually due to C. Hermite in 1881, as pointed out by Mitrinovic and Lacovic [3] in 1985. Due to this fact, most experts refer to it as Hermite-Hadamard (or sometimes, Hadamard-Hermite) inequality which is defined as follows:
Let $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ be a convex function, where $x,y\in[a,b]$ with $x<y$. Then, the following inequalities hold:

$$f\left(\frac{x+y}{2}\right)\le\frac{1}{y-x}\int_x^y f(x)\,dx\le\frac{f(x)+f(y)}{2} \tag{1}$$

The result in equation (1) is considered as a necessary and a sufficient condition for a function $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ be a convex function. This famous inequality has raised attention of the researchers of the domain of convexity and inequality theory and a variety of refinements and generalizations have been found in it. This classical Hermite-Hadamard integral inequality estimates the integral mean value of a continuous convex function $f:[a,b]\to\mathbb{R}$. The extensions of Hermite-Hadamard type integral inequalities in recent years have taken a significant growth. Tiwari and Bhatta [6] have extended the Hermite-Hadamard type integral inequality of classical convex functions to m-convex functions whose the first order derivatives are m-convex functions and some results are proved in the framework of $q-$calculus. The present paper incorporates four sections. The first section includes the concept of convex functions and its applications in different streams along with the Hermite-Hadamard integral inequality. The second section includes the definition of classical convex function as well as its extension to m-convex function. It also incorporates the result of H-H type integral inequality whose second order derivatives are convex functions. The third section highlights the main results of the research, and it concludes in the fourth section by explaining the research domain to the future researchers.

2. Preliminary Results

The concept of convex function in classical sense is defined as follows:

Definition 2.1. The function $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ is said to be convex if the following inequality

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \tag{2}$$

holds for all $x, y \in [a,b], \lambda \in [0,1]$. We say that that f is concave if $(-f)$ is convex.

G. H. Toader [5] introduced the idea of m-convexity of the function, an intermediate between the usual convexity and the star shaped property as follows:

Definition 2.2. The function $f : [0,b] \to \mathbb{R}$ is said to be m-convex function where $m \in [0,1]$ for every $x, y \in [0,b]$ and $\lambda \in [0,b]$, we have

$$f(\lambda x + m(1-\lambda)y) \leq \lambda f(x) + m(1-\lambda)f(y).$$

Remark 2.3. For $m = 1$, we recapture the concept of convex functions on $[0,b]$ and for $m = 0$, we get the concept of star shaped functions on $[0,b]$. We recall that $f : [0,b] \to \mathbb{R}$ is star shaped if

$$f(\lambda x) \leq \lambda f(x)$$

for all $\lambda \in [0,b]$ and $x \in [0,b]$.

Odzemir et al. [4] proved the following result for the case of classical convex function.

Theorem 2.4. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable mapping on $I^0, x, y \in I$ with $x < y$ and f'' be integrable on $[x,y]$, then the following equality holds:*

$$\frac{f(x)+f(y)}{2} - \frac{1}{y-x}\int_x^y f(s)\,ds = \frac{(y-x)^2}{2}\int_0^1 \lambda(1-\lambda)f''(\lambda x + (1-\lambda)y)\,d\lambda$$

3. Main Results

In this section, we extend the idea of H-H type integral inequality of second order differentiable classical convex functions to m-convex functions as follows:

Lemma 3.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable mapping on I^0 where $x, y \in I$ with $x < y$, I^0 is an interior of I and $m \in [0,1]$. If $f'' \in L([x,y])$, then the following equality holds:*

$$\frac{f(x)+f(my)}{2}-\frac{1}{my-x}\int_x^{my}f(u)\,du=\frac{(my-x)^2}{2}\int_0^1\lambda(1-\lambda)f''(\lambda x+m(1-\lambda)y)\,d\lambda$$

Proof.

$$\begin{aligned}Let\quad I_1&=\int_0^1\lambda(1-\lambda)f''(\lambda x+m(1-\lambda)y)\,d\lambda\\&=\int_0^1(\lambda-\lambda^2)f''(\lambda x+m(1-\lambda)y)\,d\lambda\end{aligned}$$

Integrating by parts, we have

$$\begin{aligned}&=\left[(\lambda-\lambda^2)\frac{f'(\lambda x+m(1-\lambda)y)}{x-my}\right]_0^1-\int_0^1(1-2\lambda)f'\frac{(\lambda x+m(1-\lambda)y)}{x-my}\,d\lambda\\&=0-\int_0^1(1-2\lambda)f'\frac{(\lambda x+m(1-\lambda)y)}{x-my}\,d\lambda\\&=\frac{1}{my-x}\int_0^1(1-2\lambda)f'(\lambda x+m(1-\lambda)y)\,d\lambda\end{aligned}$$

Again integrating by parts, we obtain

$$\begin{aligned}&=\frac{1}{my-x}\left[(1-2\lambda)\frac{f(\lambda x+m(1-\lambda)y)}{x-my}\Big|_0^1-\int_0^1 2\frac{(\lambda x+m(1-\lambda)y)}{x-my}\,d\lambda\right]\\&=\frac{1}{my-x}\left[\frac{(-1)f(x)}{x-my}-\frac{f(my)}{x-my}+\frac{2}{x-my}\int_0^1 f(\lambda x+m(1-\lambda)y)\,d\lambda\right]\\&=\frac{1}{my-x}\left[\frac{f(x)+f(my)}{my-x}-\frac{2}{my-x}\int_0^1 f(\lambda x+m(1-\lambda)y)\,d\lambda\right]\\&=\frac{f(x)+f(my)}{(my-x)^2}-\frac{2}{(my-x)^2}\int_0^1 f(\lambda x+m(1-\lambda)y)\,d\lambda\end{aligned}$$

Put $u=\lambda x+m(1-\lambda)y$. When $\lambda=0$, then $u=my$, when $\lambda=1$, then $u=x$. Also, $d\lambda=-\frac{du}{my-x}$. On substituting these values in the above relation, we get

$$\begin{aligned}\int_0^1\lambda(1-\lambda)f''(\lambda x+m(1-\lambda)y)\,d\lambda&=\frac{f(x)+f(my)}{(my-x)^2}-\frac{2}{(my-x)^2}\int_{my}^{x}f(u)\frac{(-du)}{my-x}\\\int_0^1\lambda(1-\lambda)f''(\lambda x+m(1-\lambda)y)\,d\lambda&=\frac{2}{(my-x)^2}\left[\frac{f(x)+f(my)}{2}-\frac{1}{my-x}\int_x^{my}f(u)\,du\right]\\\frac{f(x)+f(my)}{2}-\frac{1}{my-x}\int_x^{my}f(u)\,du&=\frac{(my-x)^2}{2}\int_0^1\lambda(1-\lambda)f''(\lambda x+m(1-\lambda)y)\,d\lambda.\end{aligned}$$

Remark 3.2. If $m=1$, then this result reduces to theorem 2.4.

Theorem 3.3. *Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be a twice differentiable mapping on I^0 where $x,y\in I$ with $x<y$, I^0 is*

an interior of I and $m \in [0,1]$. *If* $f'' \in L[x,y]$, *then the following inequality holds:*

$$\left|\frac{f(x)+f(my)}{2} - \frac{1}{my-x}\int_x^{my} f(u)\,du\right| \leq \frac{(my-x)^2}{64}\left(m|f''(y)|+|f''(x)|\right)$$

Proof. Using the result of 3.1 and taking modulus on both sides, we have

$$\begin{aligned}\left|\frac{f(x)+f(my)}{2} - \frac{1}{my-x}\int_x^{my} f(u)\,du\right| &= \left|\frac{(my-x)^2}{2}\int_0^1 \lambda(1-\lambda)f''(\lambda a+m(1-\lambda)y)\,d\lambda\right| \\ &\leq \frac{(my-x)^2}{2}\int_0^1 |\lambda-\lambda^2||f''(\lambda x+m(1-\lambda)y)|\,d\lambda \\ &\leq \frac{(my-x)^2}{2}\left[|f''(x)|\int_0^1 \lambda|\lambda-\lambda^2|\,d\lambda + m|f''(y)|\int_0^1 (1-\lambda)|\lambda-\lambda^2|\,d\lambda\right]\end{aligned}$$

Here

$$\int_0^1 \lambda|\lambda-\lambda^2|\,d\lambda = \int_0^{\frac{1}{2}} \lambda(\lambda-\lambda^2)\,d\lambda + \int_{\frac{1}{2}}^1 \lambda(\lambda^2-\lambda)\,d\lambda = \frac{1}{32}$$

And,

$$\int_0^1 (1-\lambda)|\lambda-\lambda^2|\,d\lambda = \int_0^{\frac{1}{2}} (1-\lambda)(\lambda-\lambda^2)\,d\lambda + \int_{\frac{1}{2}}^1 (1-\lambda)(\lambda^2-\lambda)\,d\lambda = \frac{1}{32}$$

Substituting these values, we obtain

$$\left|\frac{f(x)+f(my)}{2} - \frac{1}{my-x}\int_x^{my} f(u)\,du\right| \leq \frac{(my-x)^2}{64}\left(m|f''(y)|+|f''(x)|\right)$$

The proof is complete.

4. Conclusion

The Hermite-Hadamard integral inequality yields the lower and upper bounds of integral mean of any convex function. In this paper, we have extended the results of classical convex function into an m-convex function whose second order derivatives are m-convex functions. The interested researchers can enhance the H-H type integral inequality for other types of convex functions.

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