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Some Common Fixed Point Theorems in Fuzzy *b*-Metric Space Using Convergent Sequence

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Abstract: *As the fixed point theory has become one of the interesting field of research in pure mathematics connecting with metric space as well as fuzzy metric spaces. It has explored the new and creative ideas of research activities. One of the most significant extensions of regular metric space is fuzzy metric space. This paper's goal is to examine the notion of fuzzy b-metric space and using a pair of self mappings to establish some common fixed point results in complete fuzzy b-metric space. These mappings are used in the sequence form of function in fuzzy b-metric space and if this sequence as well as its subsequence converge at a point then this point is the unique common fixed point for these mappings. To demonstrate the strength of our primary finding, a nontrivial example is provided as part of the application. Our findings support a large number of earlier findings in the literature.*

Keywords: *Fuzzy b-metric space, Common fixed point, Cauchy sequence.*

Mathematics subject classification: 47H10, 54H25

1. Introduction

In 1965, Zadeh [16] led down an idea of fuzzy set that stands for the justification of ambiguity, imprecision and manipulation. This is more interesting and useful theory than classical set theory. These methods are applied in numerous technical and scientific fields, such as image processing, navigation and many others. In order to apply this concept in topology and analysis, many researchers have greatly expanded the theory of fuzzy sets and its applications. As the metric gives distance between any two points as a real number but it is not possible to get the exact distance between any two points in every situation. In such condition we use the idea of fuzzy metric which gives the approximate value measuring the degree of closeness between two points with respect to the given parameter such as time, temperature etc. As in 1994, George and Veeramani [7] modified the concept of fuzzy metric space which was already introduced by Kramosil and Michalek in 1975 [9]. Heilpern [8] connected the idea of fixed point theory in fuzzy metric spaces by introducing the concept of fuzzy mappings and proved some fixed point theorems for fuzzy contraction mappings in metric linear space, which is a fuzzy extension of the Banach's contraction principle. Additionally, Grabiec [6] applied fuzzy principles to the Banach Contraction Principle. Several researchers have used the idea of various compatible and contraction mappings to derive a number of fixed point theorems in fuzzy metric space.
The notion of b-metric space, which extends the concept of metric space, was initiated from the works of Bourbaki and Bakhtin [2]. Czerwik investigated several contractive mappings in b-metric space in 1993 [5]. Rectangular metric spaces were introduced by Branciari [4], who also demonstrated the

Banach- Caccippoli type fixed point result in this novel idea. In 2012, Sedghi and Shobe [14] introduced the concept of fuzzy b-metric space, which is wider and more flexible than the fuzzy metric space due to the change of triangular inequality i.e.

$$\mathbb{F}_b\left(\alpha,\beta,\frac{t}{p}\right) * \mathbb{F}_b\left(\beta,\gamma,\frac{s}{p}\right) \leq \mathbb{F}_b(\alpha,\ \gamma,\ t+s)\ \forall\ t, s > 0 \text{ and } p \geq 1.$$

In 2016, Nabadan [10] modified the idea of Sedghi and shobe and established some fixed point theorems in fuzzy b-metric space. Proposing b-rectangular metric space, in 2016, Roshan et al. [15] expanded the idea of a rectangular metric space. Additionally, he presented the idea of fuzzy quasi b-metric space. In 2024, Bhandari, Manandhar and Jha [3] introduced some fixed point results in strong fuzzy b-metric space as the generalization of fuzzy b-metric space. Ali and Hassan [1] introduced some common fixed point results in fuzzy b-metric space in 2024. In 2025, Lu et al. [11] established some results related to Cauchy sequence in fuzzy b-metric space with the answers of related questions. To prove existence of a fixed point, we can show that the sequence $\{\alpha_n\}$ is a Cauchy sequence in the fuzzy b-metric. Since the fuzzy b-metric space is assumed complete, every Cauchy sequence converges to some limit point α in the set V. Even if the entire sequence does not converge immediately, we can often extract a convergent subsequence $\{\alpha_{n_k}\}$ that converges to a common fixed point. If two different subsequences converge to different limits, then the contractive property in the fuzzy b-metric space forces those limits to coincide that ensuring uniqueness of the fixed point. Subsequences help to verify that no matter how the iteration is approached, the obtained points to the same fixed point. The convergence guarantees that the iterative process stabilizes at a point α in V. Then the continuity on f ensures that $f(\alpha) = \alpha$. Here we establish the results by using the concept of self mappings in fuzzy b-metric space and if a sequence has a convergent sub-sequence that converges at a point then this point becomes the unique common fixed point for these mappings.

2. Fundamental Concepts and Relevant Literature

Before introducing our main result, we state some basic terminologies and definitions which are frequently used in our work.

Definition 2.1. Let V be a non empty set. A mapping f: V → V is said to have a fixed point if $f(\mu) = \mu$, for μ ∈ V. In general to decide the fixed point for a given function $f(\mu) = \mu$, when a line intersects the given curve at any point or points, then such points are known as the fixed points of the line.

Definition 2.2. A pair of self mappings f, g: V → V is said to have a common fixed point if $f(\mu) = g(\mu) = \mu$ for μ ∈ V.

Definition 2.3. [12] Let $I = [0,1]$ then a mapping $* : I \times I \to I$ is said to a continuous triangular norm satisfying the properties as below:
(T1) $*$ is commutative and associative;
(T2) $*$ is continuous;
(T3) $\mu * 1 = \mu$ and $\mu * 0 = 0$, where μ lies in the unit interval [0, 1];
(T4) $(\mu * \nu) \leq (\alpha * \beta)$ for all μ, ν, α, β ∈ [0, 1] where $\mu \leq \alpha$ and $\nu \leq \beta$.
The frequently using examples of continuous t-norm are
$\mu * \nu = \mu.\nu$ (product t-norm), $\mu * \nu = \min\{\mu, \nu\}$, $\mu * \nu = \max\{\mu + \nu - 1, 0\}$ (Likasiewcz t-norm).

Definition 2.4. [7] If V be an arbitrary set, $*$ as a t-norm and a fuzzy set $\mathbb{F}$: V × V × (0, ∞) → I then a triplet (V, $\mathbb{F}$, $*$) is said to be a fuzzy metric space (FMS) satisfying the following properties:
(i) $\mathbb{F}_m(\alpha,\beta,t) > 0$;

(ii) $\mathbb{F}_m(\alpha, \beta, t) = 1$ for all t > 0 $\Leftrightarrow \alpha = \beta$;
(iii) $\mathbb{F}_m(\alpha, \beta, t) = \mathbb{F}_m(\beta, \alpha, t)$;
(iv) $\mathbb{F}_m(\alpha, \beta, t) * \mathbb{F}_m(\beta, \gamma, s) \leq \mathbb{F}_m(\alpha, \gamma, t + s)$ for all t, s > 0;
(v) $\mathbb{F}_m(\alpha, \beta, .)$: (0, ∞) → I is continuous. Where $\alpha, \beta, \gamma \in$ V and t, s > 0
The measure of closeness between α and β at any parameter t > 0 is given by $\mathbb{F}_m(\alpha, \beta, t)$.

Example 2.5. Let us consider a general metric space ($\mathbb{F}$, d) and the t-norm is taken in term of product as μ ∗ ν = μ.ν, where μ, ν are in [0, 1] and F_m be a fuzzy set on V × V → $\mathbb{R}^+$ defined as below:
$\mathbb{F}_m(\alpha, \beta, t) = \frac{\delta t^n}{\delta t^n + m d(a,b)}$, where δ, m, n are in $\mathbb{R}^+$and $\alpha, \beta \in \mathbb{F}_m$.
In such situation the triplet (V, $\mathbb{F}_m$, ∗) is known as a fuzzy metric space.
In 2012, Sedghi and Shobe [14] introduced the concept of fuzzy b-metric space, which is more wider than that of fuzzy metric spaces. Where the triangle inequality is made more flexible by using the parameter p ≥ 1. Similarly, in 2016, Nadaban [10] extended the idea of fuzzy metric space to introduce the new concept of fuzzy b-metric space connecting with idea that was introduced by Sedghi and Shobe [14] with some topological properties.
Definition 2.6. [11] Suppose $V \neq \phi$, let p ≥ 1 be a given real number and ∗ is a continuous t-norm. If we define a fuzzy set $\mathbb{F}_b$ on V × V × (0,∞) satisfying the following properties then the order triplet (V, $\mathbb{F}_b$, ∗) is said to be the fuzzy b-metric space (FbMS). ∀ $a, b, c \in$ V and t, s are positive number,
(i) $\mathbb{F}_b(\alpha, \beta, t) > 0$;
(ii) $\mathbb{F}_b(\alpha, \beta, t) = 1$ ∀ t > 0 iff $\alpha = \beta$;
(iii) $\mathbb{F}_b(\alpha, \beta, t) = \mathbb{F}_b(\alpha, \beta, t)$;
(iv) $\mathbb{F}_b(\alpha, \beta, t) * \mathbb{F}_b(\beta, \gamma, s) \leq \mathbb{F}_b(\alpha, \gamma, p(t + s))$ ∀ t, s > 0;
(v) $\mathbb{F}_b(\alpha, \beta, .)$: (0, ∞) → [0, 1] is continuous from left;
(vi) $\lim_{t \to \infty} \mathbb{F}_b(\alpha, \beta, t) = 1$
If we take p = 1 in the above definition then it becomes a FMS.

Example 2.7. Let V = [0, 1), define a mapping from $\mathbb{F}_b$: V × V × I → I defined as

$$\mathbb{F}_b(\alpha, \beta, t) = e^{-\frac{|\alpha - \beta|}{q}} \quad \forall \alpha, \beta \in V, q > 1$$

where μ ∗ ν = μ.ν. Then $\mathbb{F}_b$ is a FbMS with p = 2^{q-1}. By taking q = 2 in the given example then we can verify that the order tuple (V, $\mathbb{F}_b$, ∗) could not satisfy the property of FMS.
Definition 2.8. [8] Assume (V, $\mathbb{F}_b$, ∗) be a FbMS , a sequence $\{\alpha_n\}$ in V is said to Converge to α in V if $\lim_{n \to \infty} \mathbb{F}_b(\alpha_n, \alpha, t) = 1$ for t > 0.
Definition 2.9. [8] A sequence $\{\alpha_n\}$ in a set V is known as Cauchy sequence in a FbMS (V, $\mathbb{F}_b$, ∗) if t > 0 and m, n > n_0 then $\lim_{n \to \infty} \mathbb{F}_b(\alpha_n, \alpha_m, t) = 1$ with $n_0 \in N$.
A fuzzy b-metric space (V, $\mathbb{F}_b$, ∗) is said to be complete if every Cauchy sequence in this space is convergent.
Lemma 2.10. [13] Let (V, $\mathbb{F}_b$, ∗) be a FbMS. Then $\mathbb{F}_b(\alpha, \beta, t)$ is a non-decreasing depending with t > 0, ∀ $\alpha, \beta \in$ V .
Lemma 2.11. [13] In a FbMS (V, $\mathbb{F}_b$, ∗), if a sequence $\{g_n\}$ in V converges to g, then
(i) g is always unique. (ii) it is a Cauchy sequence.

3. Main Result

Theorem 3.1. Let (V, $\mathbb{F}_b$, ∗) be a complete FbMS with p ≥ 1 and f, g : V → V be the two mappings such that $\mathbb{F}_b(f\alpha, fg\alpha, t) \geq \mathbb{F}_b\left(\alpha, f\alpha, \frac{t}{k}\right)$ for all $\alpha \in V$, t > 0 and k ∈ $\left(0, \frac{1}{p}\right)$
Then f, and g have a unique common fixed point.

Proof.

Let $a_0 \in$ V be a fixed point and define a sequence $\{\alpha_n\}$ in V as

$f\alpha_n = \alpha_{n+1}$ when n = 0, $\alpha_1 = f\alpha$, n = 1, $\alpha = f\alpha_1$, n = 2, $\alpha_3 = f\alpha_2$, also $\alpha_2 = fg\alpha_0$ for n = 0, 1, 2, . . .

Let us assume that

$$\mathbb{F}_b(\alpha_{n+2}, \alpha_{n+1}, t) = \mathbb{F}_b(fg\alpha_n, f\alpha_n, t)$$
$$\geq \mathbb{F}_b\left(fg\alpha_n, f\alpha_n, \frac{t}{k}\right) = \mathbb{F}_b\left(\alpha_{n+2},, \alpha_{n+1}, \frac{t}{k}\right)$$

By repeating this process we obtain,

$$\mathbb{F}_b(\alpha_{n+2}, \alpha_{n+1}, t) \geq \mathbb{F}_b\left(\alpha_1, \alpha_2, \frac{t}{k^n}\right) \quad (1)$$

Hence,

$$\mathbb{F}_b(\alpha_{n+1}, \alpha_n, t) \geq \mathbb{F}_b\left(\alpha_1, \alpha_0, \frac{t}{k^n}\right) \to 1 \text{ as } n \to \infty.$$

Now it required to show $\{\alpha_n\}$ is a Cauchy sequence. For this let m, n $\geq n_0$ with m > n. Then we have

$$\mathbb{F}_b(\alpha_n, \alpha_m, t) \geq \mathbb{F}_b\left(\alpha_n, \alpha_{m-1}, \frac{t}{k}\right) * \mathbb{F}\left(\alpha_{m-1}, \alpha_m, \frac{t}{k}\right)$$

By continuing this process,

$$\mathbb{F}_b(\alpha_n, \alpha_m, t) \geq \mathbb{F}_b\left(\alpha_n, \alpha_{n+1}, \frac{t}{k^{m-n}}\right) * \cdots * \mathbb{F}_b\left(\alpha_{m-1}, \alpha_m, \frac{t}{k}\right) = 1$$

Which gives $\mathbb{F}_b(\alpha_n, \alpha_m, t) = 1$.

So, $\{\alpha_n\}$ is a Cauchy sequence as required in V.

But by hypothesis V is a complete fuzzy b-metric space, $\{\alpha_n\} \to \alpha$ in V.

So, $\lim_{n\to\infty} \alpha_n = \alpha$.

Now we have to prove that $f\alpha = \alpha$. Assume that $\alpha \neq f\alpha$, then we get

$$\mathbb{F}_b(f\alpha, \alpha, t) = \lim_{t\to\infty} \mathbb{F}_b(f\alpha, \alpha_{n+2}, t)$$
$$= \lim_{t\to\infty} \mathbb{F}_b(f\alpha, fg\alpha_n, t)$$
$$\geq \lim_{t\to\infty} \mathbb{F}_b\left(\alpha, f\alpha_n, \frac{t}{k}\right)$$
$$\geq \lim_{t\to\infty} \mathbb{F}_b\left(\alpha, f\alpha_{n+1}, \frac{t}{k}\right)$$

as t $\to \infty$, $\mathbb{F}_b(f\alpha, \alpha, t) = 1$ for all t > 0.

Hence, $f\alpha = \alpha$

Now by using the same process, if $\alpha = f\alpha \neq g\alpha$ then we have

$$\mathbb{F}_b(\alpha, g\alpha, t) = \mathbb{F}_b(f\alpha, fg\alpha, t)$$
$$\geq \mathbb{F}_b\left(\alpha, f\alpha, \frac{t}{k}\right) = 1$$

Which gives $\alpha = f\alpha = g\alpha$

That contradicts our assumption.

Using same rule, we get $\alpha = f\alpha = g\alpha$

For uniqueness:

Let $\alpha \neq \beta$ such that $\alpha = f\alpha = g\alpha$ and $\beta = f\beta = g\beta$

Assume that $\mathbb{F}_b(\alpha, \beta, t) \geq \mathbb{F}_b(f\alpha, g\beta, t) \geq \mathbb{F}_b\left(\alpha, f\alpha, \frac{t}{k}\right)$

Repeating same way, $\mathbb{F}_b(\alpha, \beta, t) \geq \mathbb{F}_b\left(\alpha, \beta, \frac{t}{k^n}\right) \to 1$ as n $\to \infty$.

This contradicts that $\alpha \neq \beta$. So, f, and g have a unique common fixed point.

Now we will verify the above theorem numerically as:

Example 3.2. Let V= $\mathbb{R}$ and the t-norm is taken as the product i.e. $\alpha * \beta = \alpha \cdot \beta$, and define a FbMS

$\mathbb{F}_b$:V× $V \times (0, \infty) \to [0,1]$ by $\mathbb{F}_b(\alpha, \beta, t) = \frac{t}{t + |\alpha - \beta|}$, t > 0 .

Since, $(\mathbb{R}, |.|)$ is complete, so $\mathbb{F}_b(\alpha,\beta,t)$ is also complete fuzzy b-metric space.
Take p=1 and define the mappings $f(\alpha)=\frac{\alpha}{2},\ \ g(\alpha)=\frac{\alpha}{3},\ \alpha\in V,$
Then as $f(\alpha)=\alpha$ gives $\frac{\alpha}{2}=\alpha\Rightarrow\alpha=0$ and $g(\alpha)=\alpha$ gives $\frac{\alpha}{3}=\alpha\Rightarrow\alpha=0.$
Hence $\alpha=0$ is the common fixed point.
To verify the contractive condition we have to show that there exists $k\in(0,1)$ such that for all $\alpha\in V$ and $t>0$, we have to verify $\quad \mathbb{F}_b(f\alpha, fg\alpha, t)\geq \mathbb{F}_b\left(\alpha, f\alpha, \frac{t}{k}\right).$

Now, $\quad f(\alpha)=\frac{\alpha}{2},\ g(\alpha)=\frac{\alpha}{3},\ fg(\alpha)=\frac{\alpha}{6},$

Thus $\quad \mathbb{F}_b(f\alpha, fg\alpha, t)=\frac{t}{t+\left|\frac{\alpha}{2}-\frac{\alpha}{6}\right|}=\frac{t}{t+|\frac{\alpha}{3}|}$

Similarly, $\mathbb{F}_b\left(\alpha, f\alpha, \frac{t}{k}\right)=\frac{\frac{t}{k}}{\frac{t}{k}+|\alpha-f(\alpha)|}=\frac{t}{t+|k\frac{\alpha}{2}|}$

Hence the inequality becomes

$$\frac{t}{t+|\frac{\alpha}{3}|}\geq\frac{t}{t+|k\frac{\alpha}{2}|}\ \Leftrightarrow\ \frac{1}{3}\leq\frac{k}{2}$$

This holds for $k\geq\frac{2}{3}$. If we choose $k=0.8\in(0,1)$, the inequality is satisfied for all $\alpha\in\mathbb{R}$ and $t>0$.
Which can be also verified by taking $a=12, t=3, k=0.8$.

Since , $\quad \mathbb{F}_b(f\alpha, fg\alpha, t)=\frac{3}{3+\frac{12}{3}}=\frac{3}{7}\approx 0.4286.$

and $\quad \mathbb{F}_b\left(\alpha, f, \frac{t}{k}\right)=\frac{3}{3+\frac{0.8\times 12}{2}}=\frac{3}{7.8}\approx 0.3846.$

Hence 0.4286≥ 0.3846.
To find the fixed point, let us define a composite function as $h=f\circ g$ then

$$h(\alpha)=fg(\alpha)=\frac{\alpha}{6}$$

Now by taking α_0=24, the sequence becomes
$\alpha_1=h(\alpha_0)$=4, $\alpha_2=\frac{2}{3}, \alpha_3=\frac{1}{9}, \alpha_4\approx 0.0185, \dots\dots\to 0.$
For any $t>0$, $\mathbb{F}_b(\alpha_n, 0, t)$=$\frac{t}{t+\alpha_n}\to 1$ as $n\to\infty$, which shows that α=0 is a unique common fixed point.

Corollary 3.3. Let (V, $\mathbb{F}_b$, $*$) be a complete fuzzy b-metric space for p ≥ 1 and f, g : V → V be two mappings such that

$$\mathbb{F}_b(f\alpha, g\beta, t)\geq\mathbb{F}_b\left(\alpha,\beta,\frac{t}{k}\right)\ \forall\ \alpha,\beta\in\text{V},\ \text{t}>0 \text{ and k}\in\left(0,\frac{1}{p}\right).$$

Then f *and* g and have a fixed point which is unique.

Theorem 3.4. Let (V, $\mathbb{F}_b$, $*$) be a complete fuzzy b-metric space and for a given real number p ≥ 1, define a sequence of function f_n : V → V such that

$$\mathbb{F}_b(f_i\alpha, f_j\beta, t)\ \geq\mathbb{F}_b\left(\alpha,\beta,\frac{t}{k}\right),\ \forall\ \alpha,\beta\in\text{V},\ \text{t}>0 \text{ and k}\in\left(0,\frac{1}{p}\right) \qquad (2)$$

Then $\{f_n\}$ has a unique common fixed point.
Proof.
Let α_0 be any fixed point of V and define a sequence $\{\alpha_n\}$ in V as
$\alpha_n=f_{n+1}\alpha_n$, for n = 0, 1, 2, . . ., then we have

$$\mathbb{F}_b(\alpha_{n+1}, a_n, \text{t})=\mathbb{F}_b(f_{n+1}\alpha_n, f\ \alpha_{n-1}, \text{t})\geq\mathbb{F}_b\left(\alpha_n,\alpha_{n-1},\frac{t}{k}\right)$$

Using induction we get,

$$\mathbb{F}_b(\alpha_{n+1}, \alpha_n, t) \geq \mathbb{F}_b\left(\alpha_1, \alpha_0, \frac{t}{k^n}\right) \rightarrow 1 \text{ as } n \rightarrow \infty.$$

Therefore $\{\alpha_n\}$ is a Cauchy sequence in V. But V is complete so $\{\alpha_n\} \rightarrow \alpha$ in V.
Now it required to show that α is a common fixed point of $\{f_n\}$. Let $f_i\alpha \neq \alpha$ for some i.
Let us consider

$$\mathbb{F}_b(f_i\alpha, \alpha, t) = \lim_{n\rightarrow\infty} \mathbb{F}_b(f_i\alpha, \alpha_{n+1}, t)$$
$$= \lim_{n\rightarrow\infty} \mathbb{F}_b(f_i\alpha, f_{n+1}\alpha_n, t).$$

Again we have, $\mathbb{F}_b(f_i\alpha, \alpha, t) \geq \mathbb{F}_b\left(\alpha, \alpha_n, \frac{t}{k^n}\right) \rightarrow 1$ as $n \rightarrow \infty$.
Thus, $\mathbb{F}_b(f_i\alpha, \alpha, t) = 1$ for each $t > 0$, which gives $f_i\alpha = \alpha$, i=1, 2, 3…n.
Hence, $f_n\alpha = \alpha$ for all n, and α is a common fixed point of $\{f_n\}$.
Again For Uniqueness:
Assume that $\alpha \neq \beta$ such that $f_i\alpha = \alpha$ and $f_j\beta = \beta$.
Consider

$$\mathbb{F}_b(\alpha, \beta, t) = \mathbb{F}_b(f_i\alpha, f_j\beta, t) \geq \mathbb{F}_b\left(\alpha, \beta, \frac{t}{k}\right)$$
$$\geq \mathbb{F}_b\left(\alpha, \beta, \frac{t}{k^n}\right) \rightarrow 1 \text{ as } n \rightarrow \infty.$$

Hence, $\mathbb{F}_b(\alpha, \beta, t) = 1$, which contradicts that $\alpha \neq \beta$. Thus, $\alpha = \beta$.

Theorem 3.5. Let (V, $\mathbb{F}_b$, $*$) be a complete fuzzy b-metric space with $p \geq 1$ and $k \in \left(0, \frac{1}{p}\right)$ where f: V → V be a mapping such that

$$\mathbb{F}_b(f\alpha, f\beta, t) \geq \mathbb{F}_b\left(\alpha, \beta, \frac{t}{k}\right) * \mathbb{F}_b\left(\alpha, f\alpha, \frac{t}{k}\right) * \mathbb{F}_b\left(\beta, f\beta, \frac{t}{k}\right) \quad (3)$$

Then f has a unique fixed point in V.

Proof.

Let α_0 is a fixed point in V and define a sequence $\{\alpha_n\}$ in V as below

$\alpha_{n+1} = f\alpha_n$ for n = 0, 1, 2,..., for $n > 0$, we can say

$$\mathbb{F}_b(\alpha_n, \alpha, t) = \mathbb{F}_b(f\alpha_{n-1}, f\alpha_n, t) \geq \mathbb{F}_b(\alpha_{n-1}, \alpha_n, t) * \cdots$$

Using induction hypothesis,

$$\mathbb{F}_b(\alpha_n, \alpha_{n+1}, t) \geq \mathbb{F}_b(\alpha_0, \alpha_1, t)$$

Hence, $\{\alpha_n\}$ is a Cauchy sequence in V. Since V is complete, $\{\alpha_n\} \rightarrow \alpha$ in V.
Now it required to show α is a fixed point of f. Suppose $\alpha \neq f\alpha$ and assume that

$$\mathbb{F}_b(\alpha, f\alpha, t) = \mathbb{F}_b(\alpha_{n+1}, f\alpha, t)$$
$$= \lim_{n\rightarrow\infty} \mathbb{F}_b(\alpha_{n+1}, f\alpha, t) = \cdots \rightarrow 1,$$

which leads to the contradicts $\alpha \neq f\alpha$. Hence, $\alpha = f\alpha$.
Now for uniqueness:
Assume $\alpha \neq \beta$ where $f\alpha = \alpha$ and $f\beta = \beta$.

$$\mathbb{F}_b(\alpha, \beta, t) = \mathbb{F}_b(f\alpha, f\beta, t) \geq \mathbb{F}_b\left(\alpha, \beta, \frac{t}{k}\right) \geq \cdots$$

So, $\mathbb{F}_b\left(\alpha, \beta, \frac{t}{k^n}\right) \rightarrow 1$ as $n \rightarrow \infty$.

$$\mathbb{F}_b(\alpha, \beta, t) = 1, \Rightarrow \alpha = \beta.$$

This leads to a contradiction. So $f\alpha = \alpha$, hence α is the fixed point of f.

Theorem 3.6. Assume (V, $\mathbb{F}_b$, $*$) be a fuzzy b-metric space, with $p \geq 1$ and define a mapping f: V → V such that $\alpha, \beta \in V$ and $\alpha \neq \beta$, $t > 0$ and $s \in \left(0, \frac{1}{p}\right)$ then

$$\mathbb{F}_b(f\alpha, f\beta, t) > min\left\{\mathbb{F}_b\left(\alpha, \beta, \frac{t}{s}\right), \mathbb{F}_b\left(\alpha, f\alpha, \frac{t}{s}\right), \mathbb{F}_b\left(\beta, f\beta, \frac{t}{s}\right)\right\}. \quad (4)$$

Let $\{\alpha_0\}$ be a point in V and if the sequence of function $\{f^n(\alpha_n)\}$ has a subsequence which converging to $\in V$, then γ is as the required fixed point of f.

Proof.

Assume $\alpha_0 \in$ V be a fixed point. Then there is $\alpha_1 \in$ V with $\alpha_1 = f\alpha_0$. Similarly, there exists $\alpha_2 \in$ V such that $\alpha_2 = f\alpha_1 = f^2\alpha_0$.
By repeating this process continuously, we obtain a sequence $\{\alpha_n\}$
where $\alpha_n = f^n\alpha_0$ for all n ≥ 1.
Let $\alpha_n = \alpha_{n+1}$ for some n. Then $\alpha_n = f\alpha_n$, and $\alpha_n = \gamma$ be a fixed point of f.
If possible assume $\alpha_n \neq \alpha_{n+1}$ ∀ n ≥ 1, so we have

$$\begin{aligned}\mathbb{F}_b(\alpha_n, \alpha_{n+1}, t) &= \mathbb{F}_b(f\alpha_{n-1}, f\alpha_n, t)\\ &> min\ \left\{\mathbb{F}_b\left(\alpha_{n-1}, \alpha_n, \frac{t}{s}\right), \mathbb{F}_b\left(f\alpha_{n-1}, f\alpha_n, \frac{t}{s}\right), \mathbb{F}_b\left(\alpha_n, f\alpha_{n-1}, \frac{t}{s}\right)\right\}\\ &= min\ \left\{\mathbb{F}_b\left(\alpha_{n-1}, \alpha_n, \frac{t}{s}\right), \mathbb{F}_b\left(\alpha_{n-1}, \alpha_n, \frac{t}{s}\right), \mathbb{F}_b\left(\alpha_n, \alpha_{n+1}, \frac{t}{s}\right)\right\}\end{aligned}$$

Thus,

$$\mathbb{F}_b(\alpha_n, \alpha_{n+1}, t) > \mathbb{F}_b\left(\alpha_{n-1}, \alpha_n, \frac{t}{s}\right), \text{ for all n} \geq 1.$$

Hence, $\{\mathbb{F}_b(\alpha_n, \alpha_{n+1}, t)\}$ is a bounded sequence which is bounded above by 1 as well as monotonically increasing also. So, it converges to some positive real number, say T.
Then, $$\lim_{n\to\infty} \mathbb{F}_b(\alpha_n, \alpha_{n+1}, t) = T$$
Since, the sequence $\{\mathbb{F}_b(\alpha_n, \alpha_{n+1}, t)\}$ has a subsequence $\{\mathbb{F}_b(\alpha_{n_k}, \alpha_{n_{k+1}}, t)\}$ which converges to T.
So, $$\lim_{n\to\infty} \mathbb{F}_b(\alpha_{n_k}, \alpha_{n_{k+1}}, t) = T$$
Now we show that T = 1, let T < 1. As $\{\alpha_n\} = \{f^n\alpha_0\}$ has a subsequence $\{\alpha_{n_k}\}$
that converges to γ so we have

$$\lim_{k\to\infty} \mathbb{F}_b(\alpha_{n_k}, \gamma, t) = 1$$

Now, $$1 > T = \lim_{k\to\infty} \mathbb{F}_b(\alpha_{n_k}, \alpha_{n_{k+1}}, t) \geq \lim_{k\to\infty} \mathbb{F}_b\left(\alpha_{n_k}, \gamma, \frac{t}{s}\right) . \mathbb{F}_b\left(\alpha, \alpha_{n_{k+1}}, \frac{t}{s}\right)$$
Then we have 1 · 1 = 1, which leads to the contradiction.
So, $$\lim_{k\to\infty} \mathbb{F}_b(\alpha_{n_k}, \alpha_{n_{k+1}}, t) = 1$$
Again we will show γ as a fixed point of f. If $\gamma \neq f\gamma$.
So we have,

$$\begin{aligned}\lim_{k\to\infty} \mathbb{F}_b(\gamma, f\gamma, t) &= \lim_{k\to\infty} \mathbb{F}_b(\alpha_{n_{k+1}}, f\gamma, t)\\ &= \lim_{k\to\infty} \mathbb{F}_b(\alpha_{n_k}, f\gamma, t)\\ &> \lim_{k\to\infty} min\left\{\mathbb{F}_b\left(\alpha_{n_k}, \gamma, \frac{t}{s}\right), \mathbb{F}_b\left(\alpha_{n_k}, f\alpha_{n_k}, \frac{t}{s}\right), \mathbb{F}_b\left(\gamma, f\alpha_{n_{k+1}}, \frac{t}{s}\right)\right\}\\ &= \min\left\{\mathbb{F}_b\left(\gamma, \gamma, \frac{t}{s}\right), \mathbb{F}_b\left(\gamma, \gamma, \frac{t}{s}\right), \mathbb{F}_b\left(\gamma, f\gamma, \frac{t}{s}\right)\right\}\\ &= \mathbb{F}_b\left(\gamma, f\gamma, \frac{t}{k}\right) \to 1 \text{ as } k \to \infty.\end{aligned}$$

that leads to the contradiction for $\gamma \neq f\gamma$. Hence, $\gamma = f\gamma$.
For uniqueness:
Assume there exists x ∈ V such that $fx = x$ and $x \neq \gamma$.
Let us assume

$$\begin{aligned}\mathbb{F}_b(\gamma, x, t) &= \mathbb{F}_b(f\gamma, fx, t)\\ &> \min\left\{\mathbb{F}_b\left(\gamma, x, \frac{t}{s}\right), \mathbb{F}_b\left(\gamma, f\gamma, \frac{t}{s}\right), \mathbb{F}_b\left(x, fx, \frac{t}{s}\right)\right\}\\ &= \min\left\{\mathbb{F}_b\left(\gamma, x, \frac{t}{s}\right), \mathbb{F}_b\left(\gamma, \gamma, \frac{t}{s}\right), \mathbb{F}_b\left(x, \gamma, \frac{t}{s}\right)\right\}\end{aligned}$$

$\mathbb{F}_b\left(\gamma, x, \frac{t}{s}\right) \to 1$ as $s \to \infty$. So, $\gamma = x$.

This leads to the contradiction. Hence, α is the fixed point as required of f.

Theorem 3.7. Assume that (V, $\mathbb{F}_b$, $*$) be a fuzzy b-metric space, with p ≥ 1 and let us define a mapping f : V → V and for all $\alpha \in V$, t > 0 and s ∈ $\left(0,\frac{1}{p}\right)$,

$$\mathbb{F}_b(f\alpha, f^2\alpha, t) > \mathbb{F}_b\left(\alpha, f\alpha, \frac{t}{s}\right).$$

For any α_0 ∈ V then the sequence $\alpha_n = f^n\alpha_0$ has a subsequence $\{\alpha_{n_k}\}$ which converges to the point γ. Then c is the unique fixed point of f.

Proof.

Let α_0 ∈ V and define a sequence by $\alpha_{n+1} = f\alpha_n = f^{n+1}\alpha_0$, for all n ≥ 1.
If we assume that for some n, $\alpha_{n+1} = \alpha_n$, then $f\alpha_n = \alpha_n$.
Thus, $\gamma = \alpha_n$ is a fixed point of f. Let $\alpha_{n+1} \neq \alpha_n$. For n ≥ 1, then we have

$$\mathbb{F}_b(\alpha_n, \alpha_{n+1}, t) = \mathbb{F}_b(f\alpha_{n-1}, f\alpha_n, t) > \mathbb{F}_b\left(\alpha_{n-1}, \alpha_n, \frac{t}{s}\right).$$

So, $\{\mathbb{F}_b(\alpha_n, \alpha_{n+1}, t)\}$ is a bounded sequence and bounded above by 1 and also monotonically increasing of positive real numbers. So, it converges to some positive real number, say T ≤ 1.
Thus, $\{\mathbb{F}_b(\alpha_n, \alpha_{n+1}, t)\}$ has a subsequence $\{\mathbb{F}_b(\alpha_{n_k}, \alpha_{n_{k+1}}, t)\}$ which converges to T.

Then, $$\lim_{k\to\infty} \mathbb{F}_b\left(\alpha_{n_k}, \alpha_{n_{k+1}}, \frac{t}{s}\right) = T.$$

Now, we will show that T = 1. If possible let us assume that T < 1. As $\{\alpha_n\}$ has a sub-sequence $\{\alpha_{n_k}\}$ which converges to γ, then we have

$$\lim_{k\to\infty} \mathbb{F}_b\left(\alpha_{n_k}, \alpha_{n_{k+1}}, \frac{t}{s}\right) = 1$$

Now, $$1 > T = \lim_{k\to\infty} \mathbb{F}_b(\alpha_{n_k}, \alpha_{n_{k+1}}, t) \geq \lim_{k\to\infty} \mathbb{F}_b\left(\alpha_{n_k}, \gamma, \frac{t}{s}\right). \mathbb{F}_b\left(\gamma, \alpha_{n_{k+1}}, \frac{t}{s}\right)$$

As both of the product approach 1, which leads to the contradiction. Hence,

$$\lim_{k\to\infty} \mathbb{F}_b\left(\alpha_{n_k}, \alpha_{n_{k+1}}, \frac{t}{s}\right) = 1$$

Now it required to show f has a fixed point as γ i.e., $\gamma = f\,\gamma$. For all t > 0, assume

$$\mathbb{F}_b(\gamma, f\gamma, t) = \lim_{k\to\infty} \mathbb{F}_b\left(\alpha_{n_{k+1}}, f\gamma, \frac{t}{s}\right) = \lim_{k\to\infty} \mathbb{F}_b\left(f\gamma_{n_k}, f\gamma, \frac{t}{s}\right)$$
$$> \lim_{k\to\infty} \mathbb{F}_b\left(\alpha_{n_k}, \gamma, \frac{t}{s}\right) = 1$$

Hence, $\mathbb{F}_b(\gamma, f\gamma, t) = 1$. So, $\gamma = f\,\gamma$.
Now for uniqueness:
Assume that $\gamma \neq$ x such that $fx = x$. Then,

$$\mathbb{F}_b(\gamma, x, t) = \mathbb{F}_b(fx, f\gamma, t) > \mathbb{F}_b(\gamma, x, t).$$

Which gives γ= x, that leads to the contradiction. Thus, γ is the fixed point of f as required.

Theorem 3.8. If (V, $\mathbb{F}_b$, $*$) be a complete fuzzy b-metric space, with p ≥ 1 and let us define two self mappings f, g : V → V with $\mathbb{F}_b(f\alpha, g\alpha, t) \geq \mathbb{F}_b(\alpha, fg\alpha, t)$ for all α∈V where $\alpha \neq f\alpha \neq g\alpha$ and for all t > 0 and s ∈ $\left(0,\frac{1}{p}\right)$. If $\alpha_0 \in V$ then there is the sequence $\{\alpha_n\} = \{gf\alpha_{n-2}\}$ for n ≥ 2 has a sub-sequence $\{\alpha_{n_k}\}$ which converges to γ, then γ is a unique common fixed point of f and g.

Proof.

Let α_0 ∈ V be an arbitrary fixed point in V . Let us define the sequence $\{\alpha_n\}$ in V as

$\alpha_1 = f\alpha_0$, $\alpha_2 = f\alpha_1, \ldots, \alpha_n = fg\alpha_{n\_2}$, for n ≥ 1, then we can say

$$\mathbb{F}_b(\alpha_{n+2}, \alpha_{n+1}, t) = \mathbb{F}_b(fg\alpha_n, gf\alpha_n, t) > \mathbb{F}_b\left(fg\alpha_n, f\alpha_n, \frac{t}{s}\right)$$
$$= \mathbb{F}_b\left(\alpha_{n+2}, \alpha_{n+1}, \frac{t}{s}\right) \to 1 \text{ as } n \to \infty.$$

Since $\mathbb{F}_b(\alpha_1, \alpha_2, t) \in [0,1]$.
Hence, $\{\mathbb{F}_b(\alpha_{n+1}, \alpha_{n+2}, t)\} > 1$ and it converges to any real number T.

So, $\lim_{n\to\infty} \mathbb{F}_b(\alpha_{n+2}, \alpha_{n+1}, t) = T$

Moreover, the sequence $\{\mathbb{F}_b(\alpha_{n+1}, \alpha_{n+2}, t)\}$ has a subsequence $\{\mathbb{F}_b(\alpha_{n_{k+2}}, \alpha_{n_{k+1}}, t)\}$ which converges to T

$$\lim_{n\to\infty} \mathbb{F}_b(\alpha_{n_{k+2}}, \alpha_{n_{k+1}}, t) = T$$

We will show that T = 1. If possible let us assume that T < 1.

Since $\{\alpha_n\}$ has a subsequence $\{\alpha_{n_k}\}$ which converges to the point c, then we have

$$\lim_{n\to\infty} \mathbb{F}_b(\alpha_{n_k}, \gamma, t) = 1$$

Now,
$$1 > T = \lim_{n\to\infty} \mathbb{F}_b(\alpha_{n_{k+1}}, \alpha_{n_{k+2}}, t)$$
$$\geq \lim_{n\to\infty} \mathbb{F}_b\left(\alpha_{n_{k+1}}, \gamma, \frac{t}{s}\right).\mathbb{F}_b\left(\alpha_{n_{k+2}}, \gamma, \frac{t}{s}\right)$$

Then the product equals 1 · 1, which leads to the contradiction. So we have T = 1.

Hence, $\lim_{n\to\infty} \mathbb{F}_b(\alpha_{n_{k+1}}, \alpha_{n_{k+2}}, t) = 1.$

Now we will show that $\gamma = f$. Let $\gamma \neq f\gamma$ and consider

$$\mathbb{F}_b(\gamma, f\gamma, t) = \lim_{n\to\infty} \mathbb{F}_b(\gamma_{n_{k+2}}, f\gamma, t) = \lim_{n\to\infty} \mathbb{F}_b(fg\alpha_{n_k}, f\gamma, t)$$
$$\geq \lim_{n\to\infty} \mathbb{F}_b(\alpha_{n_k}, \gamma, t) \geq \lim_{n\to\infty} \mathbb{F}_b(\alpha_{n_{k+1}}, \gamma, t) = 1.$$

Hence, $\mathbb{F}_b(\gamma, f\gamma, t)$=1 for each t > 0

Thus, $\gamma = f\gamma$.

Similarly, we can show that $\gamma = gf\gamma = g\gamma$ and $\gamma = gf\gamma = f\gamma$.

So, γ is a common fixed point of f and g.

For uniqueness:

Assume $\gamma \neq x$ such that $\gamma = fx = gx$. Then

$$\mathbb{F}_b(\gamma, x, t) = \mathbb{F}_b(gfx, fx, t) \geq \mathbb{F}_b(fx, x, t) \geq \mathbb{F}_b(\gamma, x, t) \to 1.$$

Hence, $\mathbb{F}_b(\gamma, x, t) = 1.$

Which leads to the contradiction that $\gamma \neq x$. So, γ = x.

Hence, x is the unique common fixed point of f and g.

Theorem 3.9. [3] Let (V, $\mathbb{F}_b$, $*$) be a complete fuzzy b-metric space such that for a given real number p ≥ 1 and k ∈ $\left(0, \frac{1}{p}\right)$, $\lim_{t\to\infty} \mathbb{F}_b(\alpha, \beta, t)$ =1

Let us define a mapping : V → V , for all $\alpha, \beta \in$ V , and

$$\mathbb{F}_b(f\alpha, f\beta, kt) \geq \mathbb{F}_b(\alpha, \beta, t) \qquad (5)$$

Then f has a unique fixed point.

4. Conclusion

In this article, by using the concept of fuzzy b-metric space introduced by Nadaban [10] which is the extended version of Sedghi and Shobe [13], we proved some common fixed point theorems in complete fuzzy b-metric space applying self mappings. Also using the concept of self mappings in convergent sequence and subsequence and if these sequences of mappings which converge at a point then this point is a unique fixed point for these mappings. Some consequences findings are also presented and few of these are extensions of previous findings in the literature. Theorem (3.1) is also verified by the numerical example. From the results discussed above, we conclude that by using the ideas of Cauchy sequence and convergent sequence in different generalized forms of fuzzy metric space, unique common fixed point can be obtained.

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Authors' Contribution

First author drafted the initial manuscript, carried out the formal analysis, prepared numerical examples, and verified the results. Second author conceived the idea of the study, designed the methodology, and supervised the overall research process and the third author contributed to the abstract, literature review, and revised it critically for intellectual content. All authors read and approved the final version of the manuscript.

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