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On Δ Operation over Fermatean Fuzzy Sets : An Application to a De-i-fuzzification Technique

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Abstract: *Fermatean fuzzy sets, a recent and advanced extension of intuitionistic fuzzy theory, expand the frontiers of uncertainty modeling by incorporating the condition that the cubic-sum of membership and non-membership degrees is bounded by one. This paper introduces the Δ operation, a novel algebraic construct tailored for Fermatean fuzzy sets, to facilitate richer comparative and transformational relationships between fuzzy evaluations. We rigorously define the Δ operation and derive several fundamental properties, including its interplay with implication and other fuzzy operations. Furthermore, we propose a specialized de-i-fuzzification method designed for the Fermatean fuzzy framework, enabling crisp representation and interpretation of fuzzy information. A key innovation of this study is the demonstrated synergy between the Δ operation and de-i-fuzzification, which permits the quantitative rectification of expert evaluation errors.*

Keywords: *Intuitionistic fuzzy sets, Pythagorean fuzzy sets, Fermatean fuzzy sets, Δ operation, De-i-fuzzification.*

1 Introduction

Fuzzy sets were introduced by Zadeh [24] in 1965. Generalizing the notion of fuzzy sets, Atanassov [2] in 1986 introduced intuitionistic fuzzy sets (IFSs). He defined some new operations over IFSs and studied several properties of such operations [3]. De, *et al.* [9] introduced several operations on IFSs and investigated their properties.

IFSs have emerged as a powerful tool for solving various decision making problems [16]. Ejegwa, *et al.* [11] discussed the application of IFSs in career determination. De, *et al.* [10] extended the Sanchez's approach of medical diagnosis to IFSs. Xiao [21] proposed a distance measure between IFSs and designed a pattern classification algorithm to address inference problems. Gohain, *et al.* [12] introduced two similarity measures between IFSs and observed their application in pattern recognition, face-mask selection and clustering problem.

It has been noticed that the totality of membership and non-membership grades is not less than or equal to one in all situations. Keeping this fact in view, Yager [22] extended the notion of IFS to Pythagorean fuzzy sets (PFSs), in which the totality of squares of membership and non-membership grades is less than or equal to one. Later, he presented an application of Pythagorean membership grades to MCDM problem [23]. In 2015, Jamkaneh and Nadarajah [14] introduced another generalization of IFS called generalized intuitionistic fuzzy sets (GIFSs) in which they considered the case that the totality of any positive integral power of membership and non-membership grades is less than or equal to one. Jamkaneh and Garg [13] introduced various new operations over GIFSs and applied them to decision making process.

Senapati and Yager [20] proposed and studied Fermatean fuzzy sets (FFSs) which are more general than IFSs and PFSs. They defined various operations over FFSs and established a Fermatean fuzzy TOPSIS method to solve an MCDM problem. In an FFS, the sum of cubes of membership and non-membership grades is less than or equal to one. Li, *et al.* [17] introduced a distance measure on FFSs and discussed its applications to pattern classifcation and multi-attribute decision-making (MADM). Buyukozkan, *et al.* [8] discussed about various extensions FFSs. Lalitha and Bhuvaneshwari [15] defined implication operation over an FFS and obtained some results.

Recently, Atanassova and Dworniczak [5] defined a new operation, *viz.* Δ over an IFS and investigated some of its properties.

Several mathematicians around the globe have been engaged in the study of de-i-fuzzification for the last three decades. Ban, *et al.* [7], Angelov [1], Atanassova and Sotirov [6] and many others worked along the de-i-fuzzification of IFSs. It is about to transforming IFSs to FSs. The important point to note here is that the above mentioned transformation reduces the degree of hesitancy or uncertainty to zero. Thus, de-i-fuzzification is a process that flattens the environment under consideration so that the degree of uncertainty of the resultant environment becomes zero. In [7] and [6], the discussed de-i-fuzzification transformation suggests to equally share the degree of hesitancy between the degrees of membership and non-membership of the IFS.

Motivation

The realm of fuzzy set theory has continuously evolved to accommodate increasing complexity and uncertainty within decision-making environments. While Intuitionistic Fuzzy Sets (IFSs) and Pythagorean Fuzzy Sets (PFSs) offer enhanced representational capabilities, they impose constraints on admissible degrees of membership and non-membership. The introduction of Fermatean Fuzzy Sets (FFSs) provides a broader and more nuanced framework for representing uncertainty and hesitation.

Despite the algorithmic richness of FFSs (as discussed in [20] and [15]), there remains a gap in the development of operations that enable sophisticated comparative analysis and error correction. Notably, the recent introduction of the Δ operation for IFSs (*see* [5]) opens avenues for more refined algebraic manipulation, yet its adaptation to the Fermatean setting remains unexplored.

Additionally, the process of de-i-fuzzification has not yet been extended for FFSs. Extending this notion to FFSs has the potential to significantly improve the reliability and interpretability of fuzzy decision support systems.

Proposed Work

This study puts forth two primary, novel contributions to the field of Fermatean fuzzy systems.

(a) **Definition and Characterization of the Δ Operation for FFSs:** For the first time, an explicit definition of the Δ operation is proposed within the Fermatean fuzzy context. The mathematical formulation respects the cubic sum constraint inherent to FFSs, ensuring that resultant sets remain valid. We rigorously analyze several algebraic and logical properties associated with Δ operation, laying a solid foundation for its theoretical and practical applications.

(b) **A New De-i-fuzzification Technique for FFSs:** Building upon classical methods that equalize the distribution of hesitancy, we introduce an original de-i-fuzzification transformation tailored for Fermatean fuzzy pairs. This innovative technique translates an FFS into a crisp outcome, effectively flattening the uncertainty environment. Furthermore, our framework elucidates a profound link between the Δ operation and the proposed de-i-fuzzification process. The synergy between these constructs is leveraged to systematically identify and correct inaccuracies in expert-generated Fermatean fuzzy pairs.

The methodology introduced herein not only advances the scheme of fuzzy set operations through the Δ operation but also delivers a robust mechanism for uncertainty reduction and error rectification. This dual advancement stands to benefit both theoretical research and practical implementations in knowledge-based expert systems.

The rest part of the paper is organized as follows. Section 2 is devoted to preliminaries in which we discuss IFSs, PFSs and FFSs. We also enlist some operations defined over FFSs. In Section 3, we introduce Δ operation over FFSs. In Section 4, we state and prove an important lemma which is useful in proving some results in the next section. In Section 5, we prove some interesting results on FFSs involving Δ operation. In Section 6, we give an example that illustrates the results numerically. In Section 7, we propose a de-i-fuzzification technique for FFSs. The proposed technique is motivated from the techniques considered in [7] and [6]. We link the de-i-fuzzification transformation with Δ operation over FFSs. Also, we discuss the application of proposed de-i-fuzzification technique through Δ operation to the rectification of incorrect Fermatean fuzzy pairs (FFPs). In Section 8, we give conclusion and discuss future work along this direction.

2 Preliminaries

This section is devoted to presenting the essential definitions, notations, and foundational ideas that serve as the basis for the theoretical analysis and results discussed in the later sections.
Let U be the universe of discourse.
An **intuitionistic fuzzy set (IFS)** on U is given by [2, 3, 4, 18, 19]

$$\mathfrak{F} = \{\langle d,\ \alpha_F(d),\ \beta_F(d)\rangle : d \in U\},$$

where $\alpha_F : U \to [0,1]$ and $\beta_F : U \to [0,1]$ are respectively the degrees of membership and non-membership of any $d \in U$ to $\mathfrak{F}$, and $(\forall d \in U)\ 0 \le \alpha_F(d) + \beta_F(d) \le 1$.
If in place of $(\forall d \in U)\ 0 \le \alpha_F(d) + \beta_F(d) \le 1$, the condition

$$(\forall d \in U)\ 0 \le \alpha_F^2(d) + \beta_F^2(d) \le 1$$

is taken into account, then $\mathfrak{F} = \{\langle d,\ \alpha_F(d),\ \beta_F(d)\rangle : d \in U\}$ is termed as a **Pythagorean fuzzy set (PFS)** (*see* [22, 23].
A **Fermatean fuzzy set (FFS)** on U is given by [20, 15]

$$\mathfrak{F} = \{\langle d,\ \alpha_F(d),\ \beta_F(d)\rangle : d \in U\},$$

where $\alpha_F : U \to [0,1]$ and $\beta_F : U \to [0,1]$ are respectively the degrees of membership and non-membership of any $d \in U$ to $\mathfrak{F}$, and $(\forall d \in U)\ 0 \le \alpha_F^3(d) + \beta_F^3(d) \le 1$. For any $d \in U$, $\pi_F(d) = \sqrt[3]{1 - \left(\alpha_F^3(d) + \beta_F^3(d)\right)}$ is called the degree of indeterminacy or the degree of hesitancy of $d \in U$ to $\mathfrak{F}$.
Clearly, the class of FFSs is more general than those of IFSs and PFSs (*see* Fig. 1).

Example 2.1. Suppose $U = \{d\}$ is the universe of discourse. Consider $\mathfrak{F} = \{\langle d,\ 0.89,\ 0.63\rangle\}$. Note that $0.89 + 0.63 = 1.52 > 1$ and $0.89^2 + 0.63^2 = 1.189 > 1$ but $0.89^3 + 0.63^3 = 0.955 \le 1$. Therefore, $\mathfrak{F} = \{\langle d,\ 0.89,\ 0.63\rangle\}$ is an FFS but neither an IFS nor a PFS on U. Thus, PFSs are worth-preferring for decision makers over IFSs and PFSs to deal with a decision-making problem.

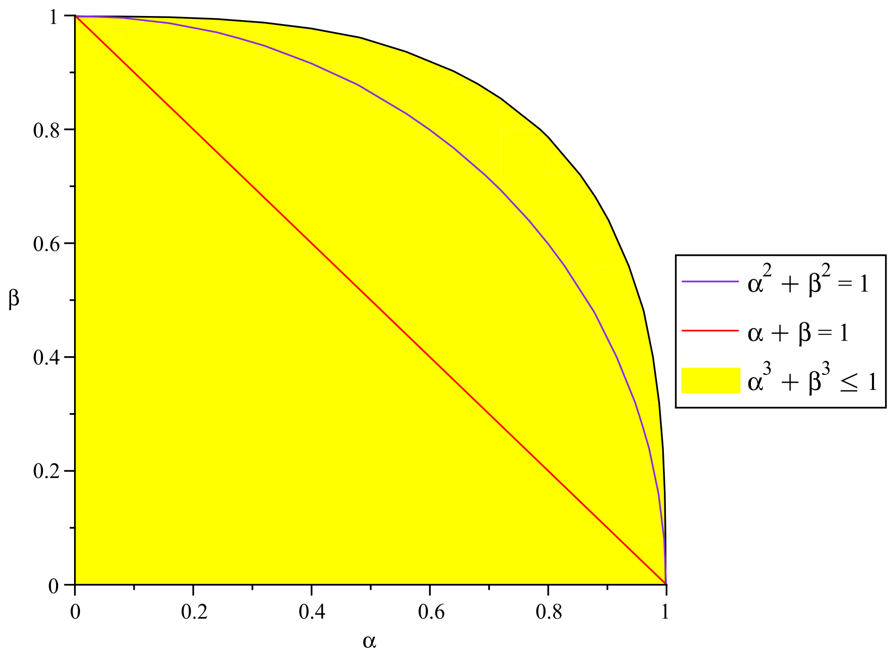

Figure 1: Region $\alpha^3 + \beta^3 \le 1$ characterizing an FFS

Henceforth, for simplicity, we shall denote an FFS $\mathfrak{P} = \{\langle d,\ \alpha_P(d),\ \beta_P(d)\rangle : d \in U\}$ by $\mathfrak{P} = (\alpha_P,\ \beta_P)$.
Let $\mathfrak{P} = (\alpha_P,\ \beta_P)$ and $\mathfrak{Q} = (\alpha_Q,\ \beta_Q)$ be two FFSs. Then (*see* [20, 15])

$$\begin{aligned}
&(i)\ \ \mathfrak{P} \cap \mathfrak{Q} := (\min(\alpha_P, \alpha_Q),\ \max(\beta_P, \beta_Q)),\\
&(ii)\ \ \mathfrak{P} \cup \mathfrak{Q} := (\max(\alpha_P, \alpha_Q),\ \min(\beta_P, \beta_Q)),\\
&(iii)\ \ \mathfrak{P}^c := (\beta_P,\ \alpha_P),\\
&(iv)\ \ \mathfrak{P} + \mathfrak{Q} := \left(\sqrt[3]{\alpha_P^3 + \alpha_Q^3 - \alpha_P^3 . \alpha_Q^3},\ \beta_P . \beta_Q\right),\\
&(v)\ \ \mathfrak{P} . \mathfrak{Q} := \left(\alpha_P . \alpha_Q,\ \sqrt[3]{\beta_P^3 + \beta_Q^3 - \beta_P^3 . \beta_Q^3}\right),\\
&(vi)\ \ \mathfrak{P} \rightharpoondown \mathfrak{Q} := (\min(\alpha_Q, \beta_P),\ \max(\alpha_P, \beta_Q)).
\end{aligned}$$

For an FFS $\mathfrak{P} = (\alpha_P,\ \beta_P)$, we define (*see* [13])

$$D_\lambda(\mathfrak{P}) := \left(\sqrt[3]{\alpha_P^3 + \lambda\, \pi_P^3},\ \sqrt[3]{\beta_P^3 + (1-\lambda)\, \pi_P^3}\right),\ \lambda \in [0,1]. \tag{1}$$

In particular, for $\lambda = 0$, we get $D_0(\mathfrak{P})$ and denote it $\Box(\mathfrak{P})$. Similarly, for $\lambda = 1$, we get $D_1(\mathfrak{P})$ and denote it $\Diamond(\mathfrak{P})$. Thus,

$$\Box(\mathfrak{P}) = \left(\alpha_P,\ \sqrt[3]{1 - \alpha_P^3}\right) \tag{2}$$

and

$$\Diamond(\mathfrak{P}) = \left(\sqrt[3]{1 - \beta_P^3},\ \beta_P\right). \tag{3}$$

3 Δ operation over FFSs

In this section, we define the Δ operation on FFSs. Its validity follows from the observation that the outcome produced is itself an FFS.
Let $\mathfrak{P} = (\alpha_P,\ \beta_P)$ and $\mathfrak{Q} = (\alpha_Q,\ \beta_Q)$ be two FFSs.
We define

$$\mathfrak{P}\Delta\mathfrak{Q} := \left(\sqrt[3]{\frac{\alpha_P^3 + \alpha_Q^3}{\alpha_P^3 + \beta_P^3 + \alpha_Q^3 + \beta_Q^3}},\ \sqrt[3]{\frac{\beta_P^3 + \beta_Q^3}{\alpha_P^3 + \beta_P^3 + \alpha_Q^3 + \beta_Q^3}}\right) \tag{4}$$

under the condition $\alpha_P^3 + \beta_P^3 + \alpha_Q^3 + \beta_Q^3 > 0$. In otherwise case, we assume that $\mathfrak{P}\Delta\mathfrak{Q} = \left(\frac{1}{\sqrt[3]{2}},\ \frac{1}{\sqrt[3]{2}}\right)$.
Note that

$$0 \le \sqrt[3]{\frac{\alpha_P^3 + \alpha_Q^3}{\alpha_P^3 + \beta_P^3 + \alpha_Q^3 + \beta_Q^3}} \le 1\,,\quad 0 \le \sqrt[3]{\frac{\beta_P^3 + \beta_Q^3}{\alpha_P^3 + \beta_P^3 + \alpha_Q^3 + \beta_Q^3}} \le 1$$

and

$$0 \le \left(\sqrt[3]{\frac{\alpha_P^3 + \alpha_Q^3}{\alpha_P^3 + \beta_P^3 + \alpha_Q^3 + \beta_Q^3}}\right)^3 + \left(\sqrt[3]{\frac{\beta_P^3 + \beta_Q^3}{\alpha_P^3 + \beta_P^3 + \alpha_Q^3 + \beta_Q^3}}\right)^3 = 1 \le 1.$$

Thus, $\mathfrak{P}\Delta\mathfrak{Q}$ is also an FFS.

Example 3.1. Suppose $U = \{d\}$ is the universe of discourse. Take two FFSs $\mathfrak{P} = (0.65, 0.87)$ and $\mathfrak{Q} = (0.88, 0.64)$. Clearly, $\alpha_P^3 + \beta_P^3 + \alpha_Q^3 + \beta_Q^3 > 0$. Using Definition (4) of Δ operation over FFSs, we get $\mathfrak{P}\Delta\mathfrak{Q} = (0.7986,\ 0.7886)$ which again falls in the region of FFSs (*see* Fig. 2).

For any positive integer $n \ge 2$, we define

$$n\mathfrak{P} = (n-1)\mathfrak{P}\,\Delta\mathfrak{P}\,.$$

Lalitha and Bhuvaneshwari [15] proved that for any two FFSs $\mathfrak{P}$ and $\mathfrak{Q}$

$$\begin{aligned}
&(i)\ \ (\mathfrak{P}^c \rightharpoondown \mathfrak{Q}) + (\mathfrak{P} \rightharpoondown \mathfrak{Q}^c)^c = \mathfrak{P} + \mathfrak{Q},\\
&(ii)\ \ (\mathfrak{P}^c \rightharpoondown \mathfrak{Q}) . (\mathfrak{P} \rightharpoondown \mathfrak{Q}^c)^c = \mathfrak{P}.\mathfrak{Q} = ((\mathfrak{P}.\mathfrak{Q})^c \rightharpoondown (\mathfrak{P} + \mathfrak{Q})).
\end{aligned} \tag{5}$$

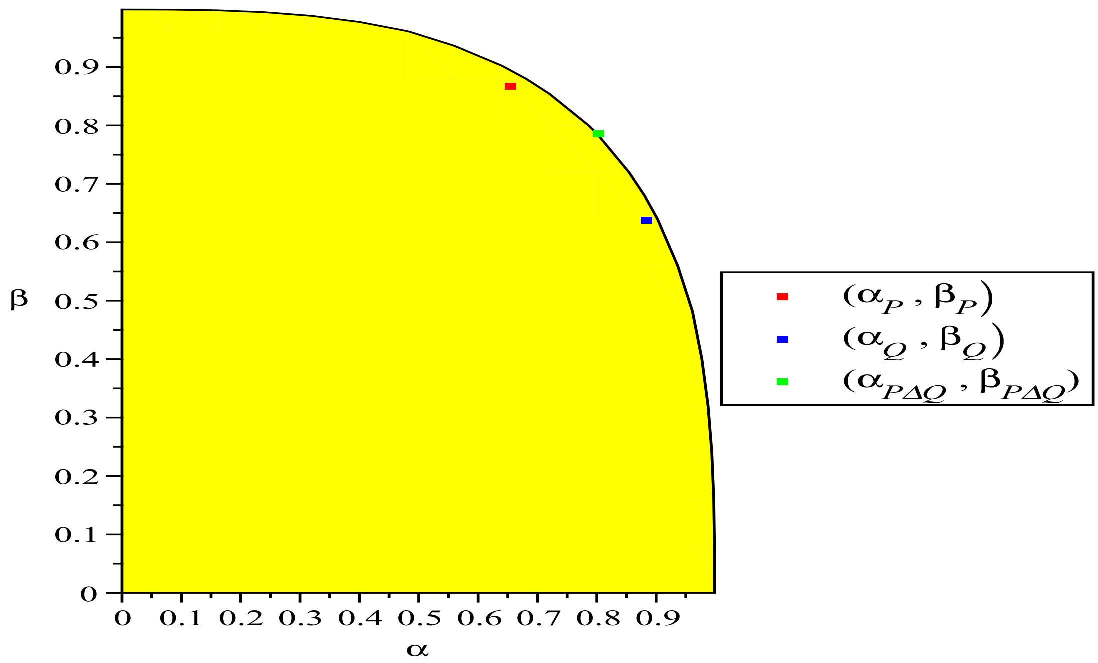

Figure 2: $\mathfrak{P}\Delta\mathfrak{Q}$ for $\mathfrak{P}=(0.65,0.87)$ and $\mathfrak{Q}=(0.88,0.64)$

4 An important Lemma

The upcoming Lemma will be employed in establishing Theorem 5.4 and Theorem 5.5 presented in the subsequent section.

Lemma 4.1. *Let $p,q,r,s\in[0,1]$ be four numbers such that $p^3+q^3+r^3+s^3>0$, $p^3+q^3\leq 1$ and $r^3+s^3\leq 1$. Then*

$$\sqrt[3]{\frac{p^3+r^3}{p^3+q^3+r^3+s^3}}\geq\sqrt[3]{\frac{p^3+r^3}{2}}, \tag{6}$$

$$\sqrt[3]{\frac{q^3+s^3}{p^3+q^3+r^3+s^3}}\leq\sqrt[3]{1-\frac{p^3+r^3}{2}}, \tag{7}$$

$$\min(p,r)\leq\sqrt[3]{\frac{p^3+r^3}{p^3+q^3+r^3+s^3}}\leq\sqrt[3]{1-\min(q^3,s^3)}\,. \tag{8}$$

Proof. For $p,q,r,s\in[0,1]$ satisfying the given conditions, it is obvious to note that

$$\frac{p^3+r^3}{p^3+q^3+r^3+s^3}\geq\frac{p^3+r^3}{2}, \tag{9}$$

which implies (6).
From (9), we can easily obtain

$$\frac{q^3+s^3}{p^3+q^3+r^3+s^3}\leq 1-\frac{p^3+r^3}{2}, \tag{10}$$

which leads to (7).
To prove (8), it suffices to show that

$$\min(p^3,r^3)\leq\frac{p^3+r^3}{p^3+q^3+r^3+s^3}\leq 1-\min(q^3,s^3)\,.$$

For $p^3 \geq r^3$, using (9), we have

$$\begin{aligned}\frac{p^3+r^3}{p^3+q^3+r^3+s^3} - \min(p^3, r^3) &\geq \frac{p^3+r^3}{2} - r^3 \\ &= \frac{p^3-r^3}{2} \\ &\geq 0.\end{aligned}$$

For $p^3 < r^3$, using (9) again, we have

$$\begin{aligned}\frac{p^3+r^3}{p^3+q^3+r^3+s^3} - \min(p^3, r^3) &\geq \frac{p^3+r^3}{2} - p^3 \\ &= \frac{r^3-p^3}{2} \\ &> 0.\end{aligned}$$

Thus,

$$\frac{p^3+r^3}{p^3+q^3+r^3+s^3} \geq \min(p^3, r^3). \tag{11}$$

Using (11), we have

$$\begin{aligned}\min(q^3, s^3) &\leq \frac{q^3+s^3}{p^3+q^3+r^3+s^3} \\ &= 1 - \frac{p^3+r^3}{p^3+q^3+r^3+s^3}.\end{aligned}$$

□

5 Some new results on FFSs involving Δ operation

In this section, we study the Δ operation on Fermatean Fuzzy Sets (FFSs) and present new findings. These results reveal key algebraic and logical properties of the Δ operation, supporting its use in fuzzy logic and uncertainty modeling.

First, we show that Δ is commutative but not associative, and that it satisfies a De Morgan-type law.

Theorem 5.1. *Let* $\mathfrak{P} = (\alpha_P,\ \beta_P)$, $\mathfrak{Q} = (\alpha_Q,\ \beta_Q)$ *and* $\mathfrak{R} = (\alpha_R,\ \beta_R)$ *be three FFSs. Then*
(i) $\mathfrak{P}\Delta\mathfrak{Q} = \mathfrak{Q}\Delta\mathfrak{P}$,
(ii) Δ *is not associative,*
(iii) $(\mathfrak{P}^c\Delta\mathfrak{Q}^c)^c = \mathfrak{P}\Delta\mathfrak{Q}$.

Proof. (i) and (ii) are straightforward from the definition of Δ.
For (iii), consider

$$\begin{aligned}(\mathfrak{P}^c\Delta\mathfrak{Q}^c)^c &= ((\beta_P,\ \alpha_P)\Delta(\beta_Q,\ \alpha_Q))^c \\ &= \left(\sqrt[3]{\frac{\beta_P^3+\beta_Q^3}{\alpha_P^3+\beta_P^3+\alpha_Q^3+\beta_Q^3}},\ \sqrt[3]{\frac{\alpha_P^3+\alpha_Q^3}{\alpha_P^3+\beta_P^3+\alpha_Q^3+\beta_Q^3}}\right)^c \\ &= \left(\sqrt[3]{\frac{\alpha_P^3+\alpha_Q^3}{\alpha_P^3+\beta_P^3+\alpha_Q^3+\beta_Q^3}},\ \sqrt[3]{\frac{\beta_P^3+\beta_Q^3}{\alpha_P^3+\beta_P^3+\alpha_Q^3+\beta_Q^3}}\right) \\ &= \mathfrak{P}\Delta\mathfrak{Q}.\end{aligned}$$

□

The following result provides a preicse form of $n\mathfrak{P}$ for an FFS $\mathfrak{P}$ and any positive integer $n \geq 2$.

Theorem 5.2. *Let $\mathfrak{P} = (\alpha_P,\ \beta_P)$ be an FFS. Then, for any positive integer $n \geq 2$*

$$n\mathfrak{P} = \left(\frac{\alpha_P}{\sqrt[3]{\alpha_P^3 + \beta_P^3}},\ \frac{\beta_P}{\sqrt[3]{\alpha_P^3 + \beta_P^3}} \right). \tag{12}$$

Proof. We apply induction on n. The statement is true for $n = 2$ as

$$\begin{aligned} 2\mathfrak{P} &= \mathfrak{P}\Delta\mathfrak{P} \\ &= \left(\sqrt[3]{\frac{\alpha_P^3 + \alpha_P^3}{\alpha_P^3 + \beta_P^3 + \alpha_P^3 + \beta_P^3}},\ \sqrt[3]{\frac{\beta_P^3 + \beta_P^3}{\alpha_P^3 + \beta_P^3 + \alpha_P^3 + \beta_P^3}} \right) \\ &= \left(\frac{\alpha_P}{\sqrt[3]{\alpha_P^3 + \beta_P^3}},\ \frac{\beta_P}{\sqrt[3]{\alpha_P^3 + \beta_P^3}} \right). \end{aligned}$$

Assume that the statement is true for any positive integer $m \geq 2$. Now,

$$\begin{aligned} (m+1)\mathfrak{P} &= m\mathfrak{P}\Delta\mathfrak{P} \\ &= \left(\frac{\alpha_P}{\sqrt[3]{\alpha_P^3 + \beta_P^3}},\ \frac{\beta_P}{\sqrt[3]{\alpha_P^3 + \beta_P^3}} \right) \Delta (\alpha_P,\ \beta_P), \text{ by induction hypothesis} \\ &= \left(\sqrt[3]{\frac{\frac{\alpha_P^3}{\alpha_P^3+\beta_P^3} + \alpha_P^3}{\frac{\alpha_P^3}{\alpha_P^3+\beta_P^3} + \alpha_P^3 + \frac{\beta_P^3}{\alpha_P^3+\beta_P^3} + \beta_P^3}},\ \sqrt[3]{\frac{\frac{\beta_P^3}{\alpha_P^3+\beta_P^3} + \beta_P^3}{\frac{\alpha_P^3}{\alpha_P^3+\beta_P^3} + \alpha_P^3 + \frac{\beta_P^3}{\alpha_P^3+\beta_P^3} + \beta_P^3}} \right) \\ &= \left(\frac{\alpha_P}{\sqrt[3]{\alpha_P^3 + \beta_P^3}},\ \frac{\beta_P}{\sqrt[3]{\alpha_P^3 + \beta_P^3}} \right), \text{ after simplification.} \end{aligned}$$

□

The following result describes the impact of modal operators $\Box$ and $\Diamond$ on the operation Δ.

Theorem 5.3. *Let $\mathfrak{P} = (\alpha_P,\ \beta_P)$ and $\mathfrak{Q} = (\alpha_Q,\ \beta_Q)$ be two FFSs. Then*

$$\begin{aligned} &(i)\ \Box(\mathfrak{P}\Delta\mathfrak{Q}) = \mathfrak{P}\Delta\mathfrak{Q}, \\ &(ii)\ \Diamond(\mathfrak{P}\Delta\mathfrak{Q}) = \mathfrak{P}\Delta\mathfrak{Q}. \end{aligned}$$

In general, for any $\lambda \in [0,1]$, $D_\lambda(\mathfrak{P}\Delta\mathfrak{Q}) = \mathfrak{P}\Delta\mathfrak{Q}$.

Proof. The result follows from the definitions of $\Box$, $\Diamond$ and $\mathfrak{P}\Delta\mathfrak{Q}$ given respectively by (2), (3) and (4). The general version can be proved using (1) and (4). □

We can observe that the modal operators $\Box$ and $\Diamond$ keep the resultant FFS fixed.

The following result shows that the Δ operation is not fully preserved by modal operators $\Box$ and $\Diamond$.

Theorem 5.4. *Let $\mathfrak{P} = (\alpha_P,\ \beta_P)$ and $\mathfrak{Q} = (\alpha_Q,\ \beta_Q)$ be two FFSs. Then*

$$\begin{aligned} &(i)\ (\Box\mathfrak{P})\Delta(\Box\mathfrak{Q}) \subseteq \Box(\mathfrak{P}\Delta\mathfrak{Q}), \\ &(ii)\ (\Diamond\mathfrak{P})\Delta(\Diamond\mathfrak{Q}) \supseteq \Diamond(\mathfrak{P}\Delta\mathfrak{Q}). \end{aligned}$$

Proof.

$$
\begin{aligned}
(i)\ \ (\Box\mathfrak{P})\Delta(\Box\mathfrak{Q}) &= \left(\alpha_P,\ \sqrt[3]{1-\alpha_P^3}\right)\Delta\left(\alpha_Q,\ \sqrt[3]{1-\alpha_Q^3}\right)\\
&= \left(\sqrt[3]{\frac{\alpha_P^3+\alpha_Q^3}{2}},\ \sqrt[3]{1-\frac{\alpha_P^3+\alpha_Q^3}{2}}\right)\\
&\subseteq \left(\sqrt[3]{\frac{\alpha_P^3+\alpha_Q^3}{\alpha_P^3+\beta_P^3+\alpha_Q^3+\beta_Q^3}},\ \sqrt[3]{\frac{\beta_P^3+\beta_Q^3}{\alpha_P^3+\beta_P^3+\alpha_Q^3+\beta_Q^3}}\right),\\
&\qquad \text{by inequalities (6) and (7) of Lemma 4.1}\\
&= \mathfrak{P}\Delta\mathfrak{Q}\\
&= \Box\left(\mathfrak{P}\Delta\mathfrak{Q}\right),\ \text{ by Theorem 5.3.}
\end{aligned}
$$

$$
\begin{aligned}
(ii)\ \ (\Diamond\mathfrak{P})\Delta(\Diamond\mathfrak{Q}) &= \left(\sqrt[3]{1-\beta_P^3},\ \beta_P\right)\Delta\left(\sqrt[3]{1-\beta_Q^3},\ \beta_Q\right)\\
&= \left(\sqrt[3]{1-\frac{\beta_P^3+\beta_Q^3}{2}},\ \sqrt[3]{\frac{\beta_P^3+\beta_Q^3}{2}}\right)\\
&\supseteq \left(\sqrt[3]{\frac{\alpha_P^3+\alpha_Q^3}{\alpha_P^3+\beta_P^3+\alpha_Q^3+\beta_Q^3}},\ \sqrt[3]{\frac{\beta_P^3+\beta_Q^3}{\alpha_P^3+\beta_P^3+\alpha_Q^3+\beta_Q^3}}\right),\\
&\qquad \text{by inequalities (6) and (7) of Lemma 4.1}\\
&= \mathfrak{P}\Delta\mathfrak{Q}\\
&= \Diamond\left(\mathfrak{P}\Delta\mathfrak{Q}\right),\ \text{ by Theorem 5.3.}
\end{aligned}
$$

□

Now, we prove a result showing that the Δ operation works as a bridge in establishing containment relationship between $(\Box\mathfrak{P})\cap(\Box\mathfrak{Q})$ and $(\Diamond\mathfrak{P})\cup(\Diamond\mathfrak{Q})$ for given FFSs $\mathfrak{P}$ and $\mathfrak{Q}$.

Theorem 5.5. *Let $\mathfrak{P} = (\alpha_P,\ \beta_P)$ and $\mathfrak{Q} = (\alpha_Q,\ \beta_Q)$ be two FFSs. Then*

$$(\Box\mathfrak{P})\cap(\Box\mathfrak{Q}) \subseteq \mathfrak{P}\Delta\mathfrak{Q} \subseteq (\Diamond\mathfrak{P})\cup(\Diamond\mathfrak{Q}).$$

Proof.

$$
\begin{aligned}
(\Box\mathfrak{P})\cap(\Box\mathfrak{Q}) &= \left(\alpha_P,\ \sqrt[3]{1-\alpha_P^3}\right)\cap\left(\alpha_Q,\ \sqrt[3]{1-\alpha_Q^3}\right)\\
&= \left(\min(\alpha_P,\alpha_Q),\ \max\left(\sqrt[3]{1-\alpha_P^3}, \sqrt[3]{1-\alpha_Q^3}\right)\right)\\
&= \left(\min(\alpha_P,\alpha_Q),\ \sqrt[3]{1-\min\left(\alpha_P^3,\alpha_Q^3\right)}\right)\\
&\subseteq \left(\sqrt[3]{\frac{\alpha_P^3+\alpha_Q^3}{\alpha_P^3+\beta_P^3+\alpha_Q^3+\beta_Q^3}},\ \sqrt[3]{\frac{\beta_P^3+\beta_Q^3}{\alpha_P^3+\beta_P^3+\alpha_Q^3+\beta_Q^3}}\right) = \mathfrak{P}\Delta\mathfrak{Q},\\
&\qquad \text{by inequalities (8) of Lemma 4.1}\\
&\subseteq \left(\sqrt[3]{1-\min\left(\beta_P^3,\beta_Q^3\right)}, \min(\beta_P,\beta_Q)\right),\\
&\qquad \text{again by inequalities (8) of Lemma 4.1}\\
&= \left(\max\left(\sqrt[3]{1-\beta_P^3}, \sqrt[3]{1-\beta_Q^3}\right), \min(\beta_P,\beta_Q)\right)\\
&= (\Diamond\mathfrak{P})\cup(\Diamond\mathfrak{Q}).
\end{aligned}
$$

□

In the subsequent results, we examine various characteristics of Fermatean Fuzzy Sets (FFSs) by applying the Δ operation in conjunction with key fuzzy set operations, namely implication, algebraic sum, algebraic product, union, and intersection.

Theorem 5.6. *Let $\mathfrak{P} = (\alpha_P,\ \beta_P)$ and $\mathfrak{Q} = (\alpha_Q,\ \beta_Q)$ be two FFSs. Then*

$$\begin{aligned}
&(i)\ \ (\mathfrak{P}^c \rightharpoondown \mathfrak{Q})\,\Delta\,(\mathfrak{P} \rightharpoondown \mathfrak{Q}^c)^c = \mathfrak{P}\Delta\mathfrak{Q},\\
&(ii)\ \ (\mathfrak{P} \rightharpoondown \mathfrak{Q})^c\,\Delta\,(\mathfrak{Q} \rightharpoondown \mathfrak{P}) = \mathfrak{P}\Delta\mathfrak{Q}^c,\\
&(iii)\ \ (\mathfrak{P}+\mathfrak{Q})\,\Delta\,(\mathfrak{P}.\mathfrak{Q}) = \mathfrak{P}\Delta\mathfrak{Q}.
\end{aligned}$$

Proof.

$$\begin{aligned}
(i)\ \ &(\mathfrak{P}^c \rightharpoondown \mathfrak{Q})\,\Delta\,(\mathfrak{P} \rightharpoondown \mathfrak{Q}^c)^c\\
&= ((\beta_P,\alpha_P) \rightharpoondown (\alpha_Q,\beta_Q))\,\Delta\,((\alpha_P,\beta_P) \rightharpoondown (\beta_Q,\alpha_Q))^c\\
&= (\min(\alpha_Q,\alpha_P),\max(\beta_P,\beta_Q))\,\Delta\,(\min(\beta_Q,\beta_P),\max(\alpha_P,\alpha_Q))^c\\
&= (\min(\alpha_P,\alpha_Q),\max(\beta_P,\beta_Q))\,\Delta\,(\max(\alpha_P,\alpha_Q),\min(\beta_P,\beta_Q))\\
&= \left(\sqrt[3]{\frac{(\min(\alpha_P,\alpha_Q))^3+(\max(\alpha_P,\alpha_Q))^3}{(\min(\alpha_P,\alpha_Q))^3+(\max(\alpha_P,\alpha_Q))^3+(\min(\beta_P,\beta_Q))^3+(\max(\beta_P,\beta_Q))^3}},\right.\\
&\qquad\left.\sqrt[3]{\frac{(\min(\beta_P,\beta_Q))^3+(\max(\beta_P,\beta_Q))^3}{(\min(\alpha_P,\alpha_Q))^3+(\max(\alpha_P,\alpha_Q))^3+(\min(\beta_P,\beta_Q))^3+(\max(\beta_P,\beta_Q))^3}}\right)\\
&= \left(\sqrt[3]{\frac{\alpha_P^3+\alpha_Q^3}{\alpha_P^3+\beta_P^3+\alpha_Q^3+\beta_Q^3}},\ \sqrt[3]{\frac{\beta_P^3+\beta_Q^3}{\alpha_P^3+\beta_P^3+\alpha_Q^3+\beta_Q^3}}\right)\\
&= \mathfrak{P}\Delta\mathfrak{Q}.
\end{aligned}$$

Similarly, one can prove (ii).
Proof of (iii) is straightforward.

□

Corollary 5.7. *For any two FFSs $\mathfrak{P} = (\alpha_P,\ \beta_P)$ and $\mathfrak{Q} = (\alpha_Q,\ \beta_Q)$*

$$((\mathfrak{P}.\mathfrak{Q})^c \rightharpoondown (\mathfrak{P}+\mathfrak{Q}))\,\Delta\,((\mathfrak{P}+\mathfrak{Q}) \rightharpoondown (\mathfrak{P}.\mathfrak{Q})^c)^c = \mathfrak{P}\Delta\mathfrak{Q}.$$

Proof. It is easy to show that

$$((\mathfrak{P}+\mathfrak{Q}) \rightharpoondown (\mathfrak{P}.\mathfrak{Q})^c)^c = \mathfrak{P}+\mathfrak{Q}. \tag{13}$$

Result follows from Part (iii) of Theorem 5.6, in view of (5 (ii)), (13) and Part (i) of Theorem 5.1. □

Theorem 5.8. *Let $\mathfrak{P} = (\alpha_P,\ \beta_P)$ and $\mathfrak{Q} = (\alpha_Q,\ \beta_Q)$ be two FFSs. Then*

$$(\mathfrak{S}\cap\mathfrak{T})\,\Delta\,(\mathfrak{S}\cup\mathfrak{T}) = \mathfrak{P}\Delta\mathfrak{Q},$$

where $\mathfrak{S} = (\mathfrak{P}^c \rightharpoondown \mathfrak{Q}) + (\mathfrak{P} \rightharpoondown \mathfrak{Q}^c)^c$ and $\mathfrak{T} = (\mathfrak{P}^c \rightharpoondown \mathfrak{Q}).(\mathfrak{P} \rightharpoondown \mathfrak{Q}^c)^c$.

Proof. From (5), we have

$$\mathfrak{S} = \mathfrak{P}+\mathfrak{Q} = \left(\sqrt[3]{\alpha_P^3+\alpha_Q^3-\alpha_P^3.\alpha_Q^3},\ \beta_P.\beta_Q\right)$$

and

$$\mathfrak{T} = \mathfrak{P}.\mathfrak{Q} = \left(\alpha_P.\alpha_Q,\ \sqrt[3]{\beta_P^3+\beta_Q^3-\beta_P^3.\beta_Q^3}\right).$$

Now

$$\mathfrak{S}\cap\mathfrak{T} = \left(\min\left(\sqrt[3]{\alpha_P^3+\alpha_Q^3-\alpha_P^3.\alpha_Q^3},\ \alpha_P.\alpha_Q\right),\ \max\left(\beta_P.\beta_Q,\ \sqrt[3]{\beta_P^3+\beta_Q^3-\beta_P^3.\beta_Q^3}\right)\right)$$

and

$$\mathfrak{S}\cup\mathfrak{T}=\left(\max\left(\sqrt[3]{\alpha_P^3+\alpha_Q^3-\alpha_P^3.\alpha_Q^3},\,\alpha_P.\alpha_Q\right),\,\min\left(\beta_P.\beta_Q,\,\sqrt[3]{\beta_P^3+\beta_Q^3-\beta_P^3.\beta_Q^3}\right)\right).$$

Also

$$\left(\min\left(\sqrt[3]{\alpha_P^3+\alpha_Q^3-\alpha_P^3.\alpha_Q^3},\,\alpha_P.\alpha_Q\right)\right)^3+\left(\max\left(\sqrt[3]{\alpha_P^3+\alpha_Q^3-\alpha_P^3.\alpha_Q^3},\,\alpha_P.\alpha_Q\right)\right)^3 = \alpha_P^3+\alpha_Q^3$$

and

$$\left(\max\left(\beta_P.\beta_Q,\,\sqrt[3]{\beta_P^3+\beta_Q^3-\beta_P^3.\beta_Q^3}\right)\right)^3+\left(\min\left(\beta_P.\beta_Q,\,\sqrt[3]{\beta_P^3+\beta_Q^3-\beta_P^3.\beta_Q^3}\right)\right)^3 = \beta_P^3+\beta_Q^3.$$

Therefore

$$\begin{aligned}&(\mathfrak{S}\cap\mathfrak{T})\Delta(\mathfrak{S}\cup\mathfrak{T})\\&=\left(\sqrt[3]{\frac{\alpha_P^3+\alpha_Q^3}{\alpha_P^3+\beta_P^3+\alpha_Q^3+\beta_Q^3}},\,\sqrt[3]{\frac{\beta_P^3+\beta_Q^3}{\alpha_P^3+\beta_P^3+\alpha_Q^3+\beta_Q^3}}\right)\\&=\mathfrak{P}\Delta\mathfrak{Q}.\end{aligned}$$

□

6 Numerical example

In this part, we present a numerical illustration to demonstrate the accuracy of our derived findings.
Let $U=\{d\}$ be the universe of discourse. Take three FFSs $\mathfrak{P}=(0.89,0.63)$, $\mathfrak{Q}=(0.6,0.9)$ and $\mathfrak{R}=(0.62,\,0.88)$. We can easily check the following.

(i) $\mathfrak{P}\Delta\mathfrak{Q}=(0.7855,0.8017)=\mathfrak{Q}\Delta\mathfrak{P}$.

(ii) $(\mathfrak{P}\Delta\mathfrak{Q})\Delta\mathfrak{R}=(0.7222,\,0.8542)\neq(0.7858,\,0.8014)=\mathfrak{P}\Delta(\mathfrak{Q}\Delta\mathfrak{R})$. This disproves the associativity of Δ operation over FFSs.

(iii) $(\mathfrak{P}^c\Delta\mathfrak{Q}^c)^c=((0.63,0.89)\Delta(0.9,0.6))^c=(0.7855,0.8017)=\mathfrak{P}\Delta\mathfrak{Q}$.

(iv) For any $n\geq 2$, $n\mathfrak{P}=(0.9038,0.6397)=\left(\frac{0.89}{\sqrt[3]{0.89^3+0.63^3}},\,\frac{0.63}{\sqrt[3]{0.89^3+0.63^3}}\right)$.

(v) $\Box(\mathfrak{P}\Delta\mathfrak{Q})=\Diamond(\mathfrak{P}\Delta\mathfrak{Q})=D_\lambda(\mathfrak{P}\Delta\mathfrak{Q})=(0.7855,0.8017)=\mathfrak{P}\Delta\mathfrak{Q}$, $\lambda\in[0,1]$.

(vi) $(\Box\mathfrak{P})\Delta(\Box\mathfrak{Q})=(0.89,0.6657)\Delta(0.6,0.9221)$
$=(0.7222,0.8141)\subseteq(0.7855,0.8017)=\mathfrak{P}\Delta\mathfrak{Q}$.

(vii) $(\Diamond\mathfrak{P})\Delta(\Diamond\mathfrak{Q})=(0.9085,0.63)\Delta(0.6471,0.9)$
$=(0.7992,0.7881)\supseteq(0.7855,0.8017)=\mathfrak{P}\Delta\mathfrak{Q}$.

(viii) $(\Box\mathfrak{P})\cap(\Box\mathfrak{Q})=(0.6,0.9221)\subseteq(0.7855,0.8017)=\mathfrak{P}\Delta\mathfrak{Q}$.

(ix) $\mathfrak{P}\Delta\mathfrak{Q}=(0.7855,0.8017)\subseteq(0.9085,0.63)=(\Diamond\mathfrak{P})\cup(\Diamond\mathfrak{Q})$.

(x) $(\mathfrak{P}^c\rightharpoondown\mathfrak{Q})\Delta(\mathfrak{P}\rightharpoondown\mathfrak{Q}^c)^c=(0.6,0.9)\Delta(0.89,0.63)=(0.7855,0.8017)=\mathfrak{P}\Delta\mathfrak{Q}$.

(xi) $(\mathfrak{P}\rightharpoondown\mathfrak{Q})^c\Delta(\mathfrak{Q}\rightharpoondown\mathfrak{P})=(0.9104,0.6260)=\mathfrak{P}\Delta\mathfrak{Q}^c$.

(xii) $(\mathfrak{P}+\mathfrak{Q})\Delta(\mathfrak{P}.\mathfrak{Q})=(0.9160,0.5670)\Delta(0.5340,0.9271)=(0.7855,0.8017)=\mathfrak{P}\Delta\mathfrak{Q}$.

(xiii) $\mathfrak{S}=(\mathfrak{P}^c\rightharpoondown\mathfrak{Q})+(\mathfrak{P}\rightharpoondown\mathfrak{Q}^c)^c=(0.9160,0.5670)$
and $\mathfrak{T}=(\mathfrak{P}^c\rightharpoondown\mathfrak{Q}).(\mathfrak{P}\rightharpoondown\mathfrak{Q}^c)^c=(0.5340,0.9271)$.

(xiv) $(\mathfrak{S}\cap\mathfrak{T})\Delta(\mathfrak{S}\cup\mathfrak{T})=(0.5340,0.9271)\Delta(0.9160,0.5670)=(0.7855,0.8017)=\mathfrak{P}\Delta\mathfrak{Q}$.

7 De-i-fuzzification of FFSs

In [6], Atanassova and Sotirov presented a de-i-fuzzification transformation in which the degree of uncertainty is equally shared between the membership and non-membership degrees. Motivated with this approach, we are going to present a version of de-i-fuzzification for FFSs in which the term $\Pi_P(x) = 1 - \sqrt[3]{1-\pi_P^3(x)}$ containing the degree of uncertainty $\pi_P(x)$ is proportionally distributed between the degrees of membership and non-membership. Let us consider the transformation of membership and non-membership degrees of a point x to an FFS $\mathfrak{P}$ given by

$$(\alpha_P(x), \beta_P(x)) \longrightarrow (\alpha_P^*(x), \beta_P^*(x)),$$

where

$$\begin{aligned} &(a) \quad \alpha_P^*(x) = \alpha_P(x) + \frac{\alpha_P(x)}{\sqrt[3]{\alpha_P^3(x)+\beta_P^3(x)}}.\Pi_P(x), \\ &(b) \quad \beta_P^*(x) = \beta_P(x) + \frac{\beta_P(x)}{\sqrt[3]{\alpha_P^3(x)+\beta_P^3(x)}}.\Pi_P(x), \end{aligned} \tag{14}$$

with

$$\Pi_P(x) = 1 - \sqrt[3]{1-\pi_P^3(x)}, \;\; \pi_P(x) = \sqrt[3]{1-\alpha_P^3(x)-\beta_P^3(x)}. \tag{15}$$

Using (15) in (14), we get $\alpha_P^*(x) = \dfrac{\alpha_P(x)}{\sqrt[3]{\alpha_P^3(x)+\beta_P^3(x)}}$ and $\beta_P^*(x) = \dfrac{\beta_P(x)}{\sqrt[3]{\alpha_P^3(x)+\beta_P^3(x)}}$. Notice that the degree of hesitancy of $(\alpha_P^*(x), \beta_P^*(x))$ is zero. Thus, the de-i-fuzzification transformation takes the form

$$(\alpha_P(x), \beta_P(x)) \longrightarrow \left(\frac{\alpha_P(x)}{\sqrt[3]{\alpha_P^3(x)+\beta_P^3(x)}}, \frac{\beta_P(x)}{\sqrt[3]{\alpha_P^3(x)+\beta_P^3(x)}} \right). \tag{16}$$

7.1 Relation between the proposed de-i-fuzzification and the Δ operation

It is clear from (16) that through the proposed de-i-fuzzification an FFS $\mathfrak{P} = (\alpha_P, \beta_P)$ transforms to $n\mathfrak{P}$, for any positive integer $n \geq 2$ (*see* Theorem 5.2). In particular, $\mathfrak{P} = (\alpha_P, \beta_P)$ transforms to $\mathfrak{P}\Delta\mathfrak{P}$, *i.e.*

$$\begin{aligned} (\alpha_P(x), \beta_P(x)) &\longrightarrow (\alpha_P(x), \beta_P(x))\,\Delta\,(\alpha_P(x), \beta_P(x)) \\ &= (\alpha_P(x), \beta_P(x))\,\Delta(0,0). \end{aligned} \tag{17}$$

This shows that the proposed de-i-fuzzification technique is very well connected with the Δ operation by (17). Therefore, we can say that Δ operation over FFSs is a kind of de-i-fuzzification operation.

7.2 Application of Δ operation through the propsed de-i-fuzzification in error correction

Any incorrect evaluation (m,n), $m, n \in [0,1]$, given by an expert can be rectified using (17) as follows:

$$(m,n)\Delta(m,n) = (m,n)\Delta(0,0) = \left(\frac{m}{\sqrt[3]{m^3+n^3}}, \frac{n}{\sqrt[3]{m^3+n^3}} \right).$$

For instance, if an expert gives an evaluation, say (0.9, 0.8), then as $0.9^3 + 0.8^3 = 1.241 > 1$, it is an incorrect FFP. Applying the de-i-fuzzification (equivalently, the Δ operation), we can obtain the rectified value $\left(\dfrac{0.9}{\sqrt[3]{0.9^3+0.8^3}}, \dfrac{0.8}{\sqrt[3]{0.9^3+0.8^3}} \right) \approx (0.8375, 0.7445)$.

Thus, our proposed de-i-fuzzification (equivalently, the Δ operation) pulls any pair outside the Fermatean region back into the region (*see* Fig. 3 and Fig. 4).

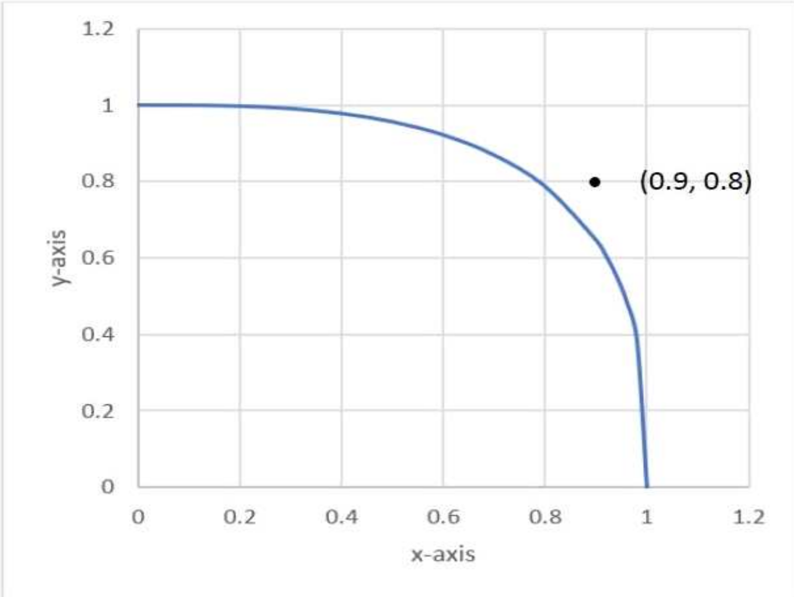

Figure 3: Incorrect FFP (before de-i-fuzzification or Δ operation)

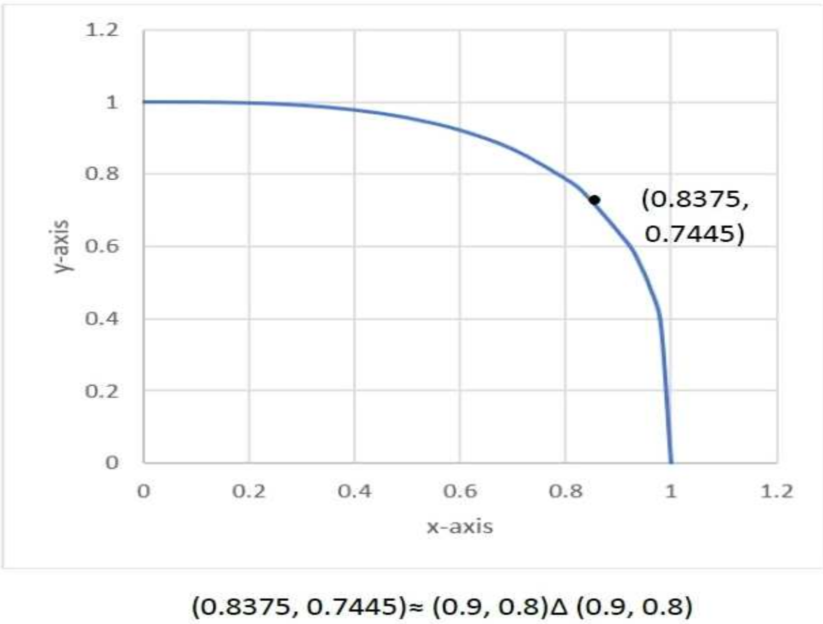

Figure 4: Correct FFP (after de-i-fuzzification or Δ operation)

8 Conclusion

The present study has expanded the operational framework of Fermatean fuzzy sets (FFSs) by introducing the Δ operation, analysing its mathematical properties, and illustrating its application in decision-making contexts. The proposed operation was proved to be symmetrical but non-associative, and it preserves the validity of the cubic-sum constraint under different transformations. An important outcome of this analysis is the finding that the

Δ operation retains its structure when subjected to the proposed de-i-fuzzification transformation, thus acting as a conceptual bridge between fuzzy information and its crisp equivalent.

The study also rvealed a close link between the Δ operation and de-i-fuzzification, enabling a unified approach both for comparative analysis and for the correction of erroneous or invalid Fermatean fuzzy pairs (FFPs). Unlike classical de-i-fuzzification, in intuitionistic fuzzy systems, which eliminates hesitation through equal redistribution, the method here proportionally allocates a term $\Pi_P(x)$ containing the degree of hesitation in accordance with the cubic-sum structure of FFSs. This ensures that even when initial data provided by an expert is inconsistent or violates the Fermatean condition, the rectification remains mathematically valid and meaningful for subsequent decision steps. The worked example confirms the practicality of the method and its capability of correcting invalid inputs by projecting them back into the admissible Fermatean region.

Future Work

Potential extensions of this research include exploring generalisations of the Δ operation to n-ary operators and adapting it to other advanced fuzzy models such as q-rung orthopair fuzzy sets. Further integration of the Δ-based de-i-fuzzification framework into established multi-criteria decision-making (MCDM) techniques, such as TOPSIS, VIKOR, or ELECTRE, could significantly enhance their performance in ranking alternatives. Another promising research avenue involves analysing the Δ operation in dynamic and time-dependent decision environments, where FFS information evolves continuously. The coupling of the proposed rectification method with optimisation models may also yield efficient strategies for minimising the adjustment required to restore invalid fuzzy information. Finally, translating these theoretical developments into decisions upport software tools would facilitate their adoption in real world applications, particularly in systems requiring close human-machine collaboration.

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