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On the Generalized Complex 2^{nd} Order Recurrences

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Abstract: *This paper investigates a generalization of complex second-order recurrence relations and develops a unified framework for their analysis. Using the Z-transform technique, we derive the corresponding generating functions and Binet-type formulas. Distinct expressions for generating functions are further obtained for the odd- and even-indexed subsequences, revealing additional structural properties. An explicit summation formula is also established, along with several fundamental identities for the resulting complex sequences. In addition, we present tridiagonal and 2×2 matrix representations of these generalized complex numbers, providing an algebraic perspective that connects recurrence relations with linear algebra. A key contribution of this work is that many well-known results for classical and generalized Fibonacci numbers appear as special cases within our broader framework, highlighting both the novelty and unifying power of the approach.*

Keywords: *Generalized complex recurrence, Z-transform technique, Odd and even indices terms, Tridiagonal matrix, Summation formula, Polar form*

1 Introduction

The Fibonacci sequence has long intrigued mathematicians for its appearance in nature and its deep mathematical connections. Studies such as [14] and [19] explore its properties, while recent generalizations [6, 20, 21, 22] have led to new and extended results in this evolving field. Adhikari [1] surveys how Fibonacci numbers and related sequences (Lucas, k-generalized, p-Fibonacci) intertwine with the golden ratio, pervade natural and human-made forms, exhibit rich recursive structures and identities, and yield important applications.
Gaussian numbers have attracted interest since Gauss's 1832 work. Generalized Gaussian Fibonacci numbers extend classical Fibonacci theory into complex numbers and algebra, offering rich insights with applications from number theory to cryptography and fractals. Parajuli et al. [16] have introduced certain sequence spaces of bi-complex numbers and examined their algebraic, topological, and geometric properties
In 1963, [12] introduced Gaussian Fibonacci numbers, extending Fibonacci sequences into the complex domain. Later works, including [13, 11, 5, 10] explored their properties, identities, and representations using recurrence relations and the complex plane. Generalizations and related sequences, such as Gaussian Jacobsthal numbers, were studied by [18, 3, 2]. Further developments, including matrix-based approaches, appear in [9, 10, 7, 4]. Recent research has expanded Gaussian Fibonacci-type sequences: [15] applied quantum calculus to Gaussian Fibonacci and Lucas quaternions, Panwar et al. [17] generalized Gaussian Fibonacci numbers through determinantal identities, and Erduvan [8] introduced Gaussian fuzzy Fibonacci numbers with corresponding formulas and classical identities.
This study presents a generalized complex Fibonacci sequence, revealing its structure through symbolic initial terms. Using the Z-transform, we derive a generalized generating function and an exact Binet formula, along with separate formulas for even and odd terms. The sequence is also expressed via tridiagonal and 2×2 matrices, and a summation formula for the first n terms is provided. The framework encompasses several known sequences, including Fibonacci, Lucas, k-Fibonacci, Pell, Jacobsthal, and their Gaussian variants.

Definition 1.1. We define the generalized 2^{nd} order complex sequence $C_n(p,q,C_1,C_2)$ by the following linear recurrence relation:

$$C_n(p,q,C_1,C_2) = pC_{n-1} + iqC_{n-2}, n \geq 3, i = \sqrt{-1} \tag{1}$$

with initial terms $C_1 = a_1 + ia_2, C_2 = b_1 + ib_2$ where a_j, b_j, p and $q\,(\neq 0)\,, j = 1,2$ are arbitrary real numbers. The expression for $C_n\,(p,q,C_1,C_2)$ is holds true [14] for every integer $n \geq 3$. For brevity, the complex sequence $C_n\,(p,q,C_1,C_2)$ will hereafter be referred to simply as C_n, many places in the text.

2 First few terms of the sequence in the generalized form

The first few terms of the sequence defined in (1) in symbolic form are:

$$C_n\,(p,q,C_1,C_2) = \left\{\begin{array}{l} C_1, C_2, pb_1 - qa_2 + i\,(pb_2 + qa_1)\,, p^2b_1 - pqa_2 - qb_2 + i\left(p^2b_2 + pqa_1 + qb_1\right), \\ p^3b_1 - p^2qa_2 - 2pqb_2 - q^2a_1 + i\left(p^3b_2 + p^2qa_1 + 2pqb_1 - q^2a_2\right), \\ p^4b_1 - p^3qa_2 - 3p^2qb_2 - 2pq^2a_1 - q^2b_1 \\ + i\left(p^4b_2 + p^3qa_1 + 3p^2qb_1 - 2pq^2a_2 - q^2b_2\right), \\ \cdots \end{array}\right\}$$

2.1 Examples

1. On substituting $a_1 = \frac{2}{3}, a_2 = \frac{1}{4}, b_1 = 2, b_2 = \frac{1}{2}, p = \frac{1}{2}, q = \frac{3}{2}$,that is $C_1 = \frac{2}{3} + i\frac{1}{4}, C_2 = 2 + i\frac{1}{2}$ in (1), we obtain

$$C_n\,(p,q,C_1,C_2) = \left\{\frac{2}{3} + \frac{1}{4}i, 2 + \frac{1}{2}i, \frac{5}{8} + \frac{5}{4}i, \frac{-7}{16} + \frac{29}{8}i, \frac{-67}{32} + \frac{11}{4}i \cdots\right\}$$

which is a complex sequence.

2. On substituting $a_1 = 0, a_2 = 0, b_1 = 1, b_2 = 0, p = 1, q = {}^1/_i$, that is $C_1 = 0, C_2 = 1$ in (1), we obtain $C_n\,(p,q,C_1,C_2) = \{0,1,1,2,3,5,8,13,21,34,55,\cdots\} = F_n$, which is the **classical Fibonacci** sequence.

3. On substituting $a_1 = 2, a_2 = 0, b_1 = 1, b_2 = 0, p = 1, q = {}^1/_i$, that is $C_1 = 2, C_2 = 1$ in (1), we obtain $C_n\,(p,q,C_1,C_2) = \{2,1,3,4,7,11,18,29,47,76,123,\cdots\} = L_n$, which is the **classical Lucas** sequence.

4. On substituting $a_1 = 1, a_2 = -1, b_1 = 2, b_2 = 1, p = 2, q = 3$,that is $C_1 = 1 - i, C_2 = 2 + i$ in (1), we obtain $C_n\,(p,q,C_1,C_2) = \{1 - i, 2 + i, 7 + 5i, 11 + 16i, 7 + 53i, -34 + 139i, -227 + 299i, -871 + 496i, \cdots\}$, which is a complex sequence.

3 Derivation of generating function for C_n

Theorem 3.1. *If $C(z)$ is the generating function of the complex sequences defined by the recursion relation (1) then*

$$C\,(z) = \frac{-C_1\,(p + iqz) - iqC_2}{1 - pz - iqz^2}, C_1 = C(0), C_2 = C(1). \tag{2}$$

where $C_1 = a_1 + ia_2$ and $C_2 = b_1 + ib_2$, $a_j, b_j, p, q\,(\neq 0) \in R$.

Proof. Using the definition (1) and the Z-transform, let $C(z)$ be the generating function of the complex sequence

$$Z\{C_n\} = C\,(z)\,. \tag{3}$$

Now apply the Z-transform to the recurrence relation (1), we have

$$Z\,(C_n) = pZ\,(C_{n-1}) + iqZ\,(C_{n-2})\,. \tag{4}$$

On employing the Z-transform shift properties, we have

$$Z\,(C_{n-1}) = zC\,(z) - C\,(0)$$
$$Z\,(C_{n-2}) = z^2C\,(z) - zC\,(0) - C\,(1)$$

Substitute these into the relation in (3) and from (4), we obtain

$$C(z) = p[zC(z) - C(0)] + iq\left[z^2C(z) - zC(0) - C(1)\right]$$

Collecting terms involving $C(z)$ and simplify, we have

$$\left(1 - pz - iqz^2\right)C(z) = -C(0)(p + iqz) - iqC(1)$$

Simplify the above expression, we obtain

$$C(z) = \frac{-C_1(p + iqz) - iqC_2}{1 - pz - iqz^2}, C_1 = C(0), C_2 = C(1).$$

Hence the result. □

4 Derivation of Binet type formula for C_n

Theorem 4.1. *For the generalized complex recurrence relation (1), the n^{th} generalized number is*

$$C_n = \left(\frac{z_1^n - z_2^n}{z_1 - z_2}\right)C_2 - \left(\frac{z_2 z_1^n - z_1 z_2^n}{z_1 - z_2}\right)C_1, \tag{5}$$

where

$$z_{1,2} = \frac{p \pm \sqrt{p^2 + 4iq}}{2iq}.$$

Proof. Since the generating function $C(z)$ (2) of linear recurrence relation (1), is:

$$C(z) = \frac{-C(0)(p + iqz) - iqC(1)}{1 - pz - iqz^2}, C_1 = C(0), C_2 = C(1).$$

The denominator of above relation is a quadratic, which is factored as

$$1 - pz - iqz^2 = (z - z_1)(z - z_2),$$

where

$$z_{1,2} = \frac{p \pm \sqrt{p^2 + 4iq}}{2iq}.$$

Decompose the generating function into partial fractions, we have

$$C(z) = \frac{-C(0)(p + iqz) - iqC(1)}{1 - pz - iqz^2}. \equiv \frac{P}{(z - z_1)} + \frac{Q}{(z - z_2)},$$

where P and Q are constants determined by the initial conditions $C(0)$ and $C(1)$.

$$P = \frac{C_2 - C_1 z_2}{z_1 - z_2},\ Q = \frac{C_1 z_1 - C_2}{z_1 - z_2}$$

Since $Z(z_1^n) = \frac{1}{z - z_1}$ and $Z(z_2^n) = \frac{1}{z - z_2}$ (the inverse Z -transform yields the Binet-type formula:

$$C_n = Pz_1^n + Qz_2^n. \tag{6}$$

On Substituting P and Q into (6), we obtain

$$C_n = \left(\frac{z_1^n - z_2^n}{z_1 - z_2}\right)C_2 - \left(\frac{z_2 z_1^n - z_1 z_2^n}{z_1 - z_2}\right)C_1.$$

This is the Binet type formula (5) for generalized complex sequences (1) . □

5 Identities for complex Generalized Fibonacci numbers

Theorem 5.1. *For an integer the generalized complex recurrence relation (1),*

$$C_{-m} = -(z_1 z_2)^{-m}\left[\left(\frac{z_1^m - z_2^m}{z_1 - z_2}\right)C_2 - \left(\frac{z_1^{m+1} - z_2^{m+1}}{z_1 - z_2}\right)C_1\right].$$

Theorem 5.2 (Cassini identity)**.** *For an integer m, the generalized complex recurrence relation (1), the Cassini identity is*

$$C_{n+1}C_{n-1} - C_n^2 = (-iq)^{n-1}\left(pC_1C_2 - C_2^2 + iC_1^2\right)$$

where $pC_1C_2 - C_2^2 + iC_1^2 = \left\{\begin{array}{l}\left[(a_1b_1 - a_2b_2)\,p - a_1a_2q - b_1^2 + b_2^2\right] \\ +\,i\left[(a_1b_2 + a_2b_1)\,p - 2b_1b_2 - \left(a_1^2 + a_2^2\right)q\right]\end{array}\right\}$

Theorem 5.3 (Catalan's identity)**.** *For the generalized complex recurrence relation (1), the Catalan'sidentity is*

$${C_n}^2 - C_{n+r}C_{n-r} = (-iq)^{n-r}\left(\frac{z_1^n - z_2^n}{z_1 - z_2}\right)^2\left(pC_1C_2 - C_2^2 + iC_1^2\right).$$

Theorem 5.4 (D'Ocagne'sidentity)**.** *For the generalized complex recurrence relation (1), the D'Ocagne's identityis*

$$C_{n+1}C_{n-1} - C_n^2 = (-iq)^{n-1}\left(pC_1C_2 - C_2^2 + iC_1^2\right).$$

Theorem 5.5 For the generalized complex recurrence relation (1), the identity is

$$C_mC_{n+1} - C_nC_{m+1} = (-iq)^n\left(\frac{z_1^{m-n} - z_2^{m-n}}{z_1 - z_2}\right)\left(C_2^2 - pC_1C_2 - iqC_1^2\right).$$

Theorem 5.5. *For the generalized complex recurrence relation (1),*

$$C_{n+1}C_{n+r} - C_nC_{n+r+1} = -(-iq)^n\left(\frac{z_1^r - z_2^r}{z_1 - z_2}\right)\left(C_2^2 - pC_1C_2 - iqC_1^2\right)).$$

Theorem 5.6. *For the generalized complex recurrence relation (1),*

$$C_{n+a}C_{n+b} - C_nC_{n+a+b} = (-iq)^n\left(\frac{z_1^a - z_2^a}{z_1 - z_2}\right)\left(\frac{z_1^b - z_2^b}{z_1 - z_2}\right)\left(C_2^2 - pC_1C_2 - iqC_1^2\right).$$

6 Generating function for even $C_{E_n} = C_{2n},\ n \geq 1$ and odd terms $C_{O_n} = C_{2n-1},\ n \geq 1$

6.1 Generating function for even terms $C_{E_n} = C_{2n},\ n \geq 1$

Theorem 6.1. *If* $C_E(w)$ *is generating function for the even terms of the complex sequence defined by the recursion (1)*

$$C_E(w) = \frac{\left[-\left(p^2 + ipq\right)C_{E_1} - iqC_{E_2}\right]w - q^2C_{E_2}}{w^2 - (p^2 + 2iq)\,w - q^2} \tag{7}$$

$C_{E_1} = b_1 + ib_2 = C_2$ *and* $C_{E_2} = \left(p^2b_1 - qb_2 - pqa_2\right) + i\left(p^2b_2 + qb_1 + pqa_1\right) = C_4.$

Proof. Using the definition (2) of generating function of generalized complex Fibonacci sequences,

$$C_E(w) = \frac{C(\sqrt{w}) + C(-\sqrt{w})}{2}$$

is the generating function of even terms the generalized complex Fibonacci sequences, Simplify $C_E(w)$, we obtain

$$C_E(w) = \frac{\left[-\left(p^2 + ipq\right)C_{E_1} - iqC_{E_2}\right]w - q^2C_{E_2}}{w^2 - (p^2 + 2iq)\,w - q^2}.$$

Hence the result. □

6.2 Generating function for odd terms $C_{O_n} = C_{2n-1},\ n \geq 1$

Theorem 6.2.1 If $C_O(w)$ is generating function for the odd terms of the complex sequence defined by the recursion (1) then

$$C_O(w) = \frac{-(p+iq)C_{O_1}w + \left(ipq - q^2\right)C_{O_1} - ipqC_{O_2}}{w^2 - (p^2+2iq)w - q^2}. \tag{8}$$

where $C_{O_1} = a_1 + ia_2 = C_1$ and $C_{O_2} = (pb_1 - qa_2) + i(pb_2 + qa_1) = C_3$.

Proof. Using the definition (2) of generating function of generalized complex Fibonacci sequences,

$$C_O(w) = \frac{C(\sqrt{w}) - C(-\sqrt{w})}{2\sqrt{w}}$$

is the generating function of odd terms the generalized complex Fibonacci sequences, Simplify $C_O(w) = \frac{C(\sqrt{w}) - C(-\sqrt{w})}{2\sqrt{w}}$, we obtain

$$C_O(w) = \frac{-(p+iq)C_{O_1}w + \left(ipq - q^2\right)C_{O_1} - ipqC_{O_2}}{w^2 - (p^2+2iq)w - q^2}.$$

Hence the result. □

7 Explicit Binet type formuls for even and odd indices ($C_{E_n} = C_{2n},\ ,\ C_{O_n} = C_{2n-1}, n \geq 1$)

7.1 Binet type formula for terms at even indices ($C_{E_n} = C_{2n},\ n \geq 1$)

Theorem 7.1. *If C_E is generating function for the even terms of the complex sequence defined in (7), then Bimet formula for complex sequence CE_n is*

$$C_{E_n} = \left(\frac{w_1^n - w_2^n}{w_1 - w_2}\right)C_{E_2} - \left(\frac{w_2w_1^n - w_1w_2^n}{w_1 - w_2}\right)C_{E_1}. \tag{9}$$

where $C_{E_1} = b_1 + ib_2 = C_2$ and $C_{E_2} = \left(p^2b_1 - qb_2 - pqa_2\right) + i\left(p^2b_2 + qb_1 + pqa_1\right) = C_4$. and w_1, w_2 are roots of $w^2 - \left(p^2 + 2iq\right)w - q^2 = 0$ given by

$$w_{1,2} = \frac{\left(p^2+2iq\right) \pm \sqrt{\left(p^2+2iq\right)^2 + 4q^2}}{2}$$

Proof. The generating function (7) for the even terms of the generalized complex Fibonacci sequence is:

$$C_E(w) = \frac{\left[-\left(p^2+ipq\right)C_{E_1} - iqC_{E_2}\right]w - q^2C_{E_2}}{w^2 - (p^2+2iq)w - q^2}$$

The denominator of it is a quadratic, which can be written as

$$w^2 - \left(p^2+2iq\right)w - q^2 = 0$$

where

$$w_{1,2} = \frac{\left(p^2+2iq\right) \pm \sqrt{\left(p^2+2iq\right)^2 + 4q^2}}{2}$$

For a second-order linear recurrence relation with characteristic roots w_1 and w_2 the general solution follows the form:

$$C_{E_n}(w) = Rw_1^n + Sw_2^n. \tag{10}$$

We decompose (7) into partial fractions:

$$C_E(z) = \left[-\left(p^2+ipq\right)C_{E_1} - iqC_{E_2}\right]z - q^2CE_2 \equiv \frac{R}{z-z_1} + \frac{S}{z-z_2}$$

Here R and S are constants, determined from initial conditions. Using the generating function structure, we extract two known values of C_{E_n}. Multiplying both sides by the denominator, we obtain

$$R(z-z_2)+S(z-z_1)=\left[-\left(p^2+ipq\right)C_{E_1}-iqCE_2\right]z-q^2C_{E_2}.$$

Taking $w=w_1$ and $w=w_2$, we can solve for R and S

$$R=\frac{C_{E_2}-w_2C_{E_1}}{w_1-w_2},S=\frac{-\left(C_{E_2}-w_1C_{E_1}\right)}{w_1-w_2}$$

Substitute R and S in (10), we have

$$C_{E_n}=\left(\frac{w_1^n-w_2^n}{w_1-w_2}\right)C_{E_2}-\left(\frac{w_2w_1^n-w_1w_2^n}{w_1-w_2}\right)C_{E_1}.$$

This is the Binet type formula Formula for even terms indices of the generalized complex Fibonacci sequences. □

7.2 Binet type formula for odd terms indices ($C_{O_n}=C_{2n-1},\ n\geq 1$)

Theorem 7.2. *If C_{odd} is generating function of the odd terms of the complex sequence defined in (8), then Bimet formula for complex sequence C_{O_n} is*

$$C_{O_n}=\left(\frac{w_1^n-w_2^n}{w_1-w_2}\right)C_{O_1}-\left(\frac{w_2w_1^n-w_1w_2^n}{w_1-w_2}\right)C_{O_2} \tag{11}$$

where $C_{O_1}=a_1+ia_2=C_1$ and $C_{O_2}=(pb_1-qa_2)+i(pb_2+qa_1)=C_3$.

Proof. The generating function (8) for the odd terms indices of the generalized complex Fibonacci sequence is:

$$C_O(w)=\frac{-(p+iq)C_{O_1}w+\left(ipq-q^2\right)C_{O_1}-ipqC_{O_2}}{w^2-(p^2+2iq)w-q^2}.$$

The denominator of it is a quadratic, which can be written as

$$w^2-\left(p^2+2iq\right)w-q^2=0$$

where

$$w_{1,2}=\frac{\left(p^2+2iq\right)\pm\sqrt{\left(p^2+2iq\right)^2+4q^2}}{2}.$$

For a second-order linear recurrence relation with characteristic roots w_1 and w_2 the general solution follows the form:

$$C_{O_n}(w)=Pw_1^n+Qw_2^n. \tag{12}$$

We decompose (8) into partial fractions:

$$C_O(w)=\frac{-(p+iq)C_{O_1}w+\left(ipq-q^2\right)C_{O_1}-ipqC_{O_2}}{(w-w_1)(w-w_2)}\equiv\frac{P}{(w-w_1)}+\frac{Q}{(w-z_w)}.$$

Here P and Q are constants to be determined from initial conditions. Using the generating function structure, we extract two known values of C_{O_n}.

Multiplying both sides by the denominator, we obtain

$$P(z-z_2)+Q(z-z_1)=-(p+iq)zC_{O_1}+\left(ipq-q^2\right)C_{O_1}-ipqC_{O_2}.$$

Taking $w=w_1$ and $w=w_2$, we can solve for P and Q

$$P=\frac{C_{O_2}-w_2C_{O_1}}{w_1-w_2},Q=\frac{-\left(C_{O_2}-w_1C_{O_1}\right)}{w_1-w_2}$$

Substitute P and Q in (14), we have

$$C_{O_n}=\left(\frac{w_1^n-w_2^n}{w_1-w_2}\right)C_{O_1}-\left(\frac{w_2w_1^n-w_1w_2^n}{w_1-w_2}\right)C_{O_2}.$$

This is the Binet type formula formula for odd terms indices of the generalized complex Fibonacci sequences. □

Example 7.3. Using (5) (9) and (11), we generate odd and even indices terms

Table 1: Example Odd and Even Terms indices

In $C_1 = a_1 + ia_2, C_2 = b_1 + ib_2$, take $a_1 = 2, a_2 = 0, b_1 = 1, b_2 = 0$, and $p = 3, q = 2$			
Terms (n)	Generalized terms $C_n(p,q,C_1,C_2)$	Odd indices terms $C_{O_n}(p,q,C_1,C_2)$	Even indices terms $C_{E_n}(p,q,C_1,C_2)$
0	2.000-0.000i	1.000-0.000i	2.000-0.000i
1	1.000-0.000i	9.000+14.000i	3.000+4.000i
2	3.000+4.000i	29.000+162.000i	19.000+48.000i
3	9.000+14.000i	-351.000+1.630e3i	-9.000+524.000i
4	19.000+48.000i	-9.563e3+1.391e4i	-2.101e3+4.872e3i
5	29.000+162.000i	-1.431e5+9.349e4i	-3.843e4+3.754e4i
6	-9.000+524.000i	-1.700e6+3.246e5i	-5.045e5+2.036e5i
7	-351.000+1.630e3i	-1.717e7-3.506e6i	-5.508e6-3.514e4i
8	-2.101e3+4.872e3i	-1.473e8-9.895e7i	-5.145e7-2.154e7i
9	-9.563e3+1.391e4i	-9.990e8-1.494e9i	-3.990e8-3.998e8i
10	-3.843e4+3.754e4i	-3.604e9-1.784e10i	-2.197e9-5.280e9i

8 Summation formula Complex Fibonacci Numbers

Theorem 8.1. *For the generalized Complex recurrence relation (5), the n^{th} paritial sum of generalized Complex Fibonacci Numbers is*

$$S_n = \frac{1}{p+iq-1}\left[C_{n+1} + iqC_n - C_2 + (p-1)C_1\right]. \tag{13}$$

Proof. Employing the Binet formula defined in (5), and the sum upto the first n terms is denoted by S_n, then

$$\begin{aligned} S_n &= \sum_{n=0}^{n}\left[\left(\frac{z_1^n - z_2^n}{z_1 - z_2}\right)C_2\right] - \sum_{n=0}^{n}\left(\frac{z_2 z_1^n - z_1 z_2^n}{z_1 - z_2}\right)C_1 \\ &= \frac{1}{(z_1 - z_2)}\sum_{n=0}^{n}(z_1^n - z_2^n)C_2 - \frac{1}{(z_1 - z_2)}\sum_{n=0}^{n}(z_2 z_1^n - z_1 z_2^n)C_1 \end{aligned} \tag{14}$$

Now

$$\begin{aligned} \sum_{n=0}^{n}(z_1^n - z_2^n)C_2 &= \left[\left(1 + z_1 + z_1^2 + \cdots z_1^n\right) - \left(1 + z_2 + z_2^2 + \cdots z_2^n\right)\right]C_2 \\ &= \left[\left(\frac{z_1^{n+1} - 1}{z_1 - 1}\right) - \left(\frac{z_2^{n+1} - 1}{z_2 - 1}\right)\right]C_2 \\ &= \frac{1}{(z_1 + z_2) - z_1 z_2 - 1}\left[\left(z_1^{n+1} - z_2^{n+1}\right) - z_1 z_2\left(z_1^n - z_2^n\right) - (z_1 - z_2)\right]C_2. \end{aligned} \tag{15}$$

$$\begin{aligned} \sum_{n=0}^{n}(z_2 z_1^n - z_1 z_2^n)C_1 &= \left[\left(\frac{z_1^{n+1} - 1}{z_1 - 1}\right)z_2 - \left(\frac{z_2^{n+1} - 1}{z_2 - 1}\right)z_1\right] \\ &= \frac{1}{(z_1 + z_2) - z_1 z_2 - 1}\begin{bmatrix} \left(z_1^{n+1} z_2 - z_2^{n+1} z_1\right) - z_1 z_2\left(z_1^n z_2 - z_2^n z_1\right) \\ -(z_1 - z_2)(z_1 + z_2 - 1) \end{bmatrix}C_1. \end{aligned} \tag{16}$$

Now combining (14)-(16), we obtain

$$= \frac{1}{(z_1 + z_2) - z_1 z_2 - 1}\begin{bmatrix} \left(\frac{z_1^{n+1} - z_2^{n+1}}{z_1 - z_2}\right)C_2 - \left(\frac{z_1^{n+1} z_2 - z_2^{n+1} z_1}{z_1 - z_2}\right)C_1 \\ -z_1 z_2\left[\left(\frac{z_1^n - z_2^n}{z_1 - z_2}\right)C_2 - \left(\frac{z_1^n z_2 - z_2^n z_1}{z_1 - z_2}\right)C_1\right]. \\ -C_2 + (z_1 + z_2 - 1)C_1 \end{bmatrix}.$$

On simplification

$$S_n = \frac{1}{(z_1+z_2)-z_1z_2-1}\left[C_{n+1}-z_1z_2C_n-C_2+(z_1+z_2-1)C_1\right],$$

which can be written as

$$S_n = \frac{1}{p+iq-1}\left[C_{n+1}-z_1z_2C_n-C_2+(p-1)C_1\right],$$

where $z_1+z_2=p,\ z_1z_2=-iq$. Hence the result. □

9 Complex Generalized sequences: Special cases

1. **k-Complex Fibonacci numbers sequences** If we take $p=k,\ q=1,\ a_1=0,\ a_2=0,\ b_1=1,\ b_2=0$ in the equation (1), then we get the Gaussian k- Fibonacci sequence $C_n(k,C_1,C_2)$ by the following linear complex recurrence relation:

$$C_n(k,C_1,C_2) = kC_{n-1}+iC_{n-2}, n\geq 3, \tag{17}$$

with initial terms $C_1=a_1+ia_2=0$, $C_2=b_1+ib_2=1$, and $i=\sqrt{-1}$, $a_j,b_j,k\in R$, $(j=1,2)$ are arbitrary non–zero real numbers.

Generating Function

$$C(z) = \frac{-i}{z^2-kz-i}.$$

Binet Formula

$$C_n = \left(\frac{z_1^n-z_2^n}{z_1-z_2}\right)$$

where $z_{1,2}=\frac{k\pm\sqrt{k^2+4i}}{2}$ roots of the characteristic equation $z^2-kz-i=0$.

Explicit Summation formula

$$S_n = \frac{1}{k+i-1}\left[C_{n+1}-z_1z_2C_n+(k-2)\right],$$

where $z_1+z_2=k,\ z_1z_2=-i.$

Remark The above results are the corresponding results for the Generalized complex k-Fibonacci sequences. If we take $k=1,2,3$ in the above Gaussian k- Fibonacci sequence, then we obtain the results for the Generalized complex Fibonacci, complex Pell sequence and complex Fibonacci sequence of order 3.

2. **Lucas sequence** If we take $p=1,\ q=1,\ a_1=2,\ a_2=0,\ b_1=1,\ b_2=0$ in the equation (1), then we get the Lucas sequence $C_n(C_1,C_2)$ by the following linear complex recurrence relation:

$$C_n(C_1,C_2) = C_{n-1}+iC_{n-2}, n\geq 3, \tag{18}$$

with initial terms $C_1=a_1+ia_2=2$, $C_2=b_1+ib_2=1$, and $i=\sqrt{-1}$, $a_j,b_j,\in R$, $(j=1,2)$ are arbitrary non–zero real numbers.

Generating Function

$$C(z) = \frac{-2z(1+i)-i}{z^2-z-i}.$$

Binet Formula

$$C_n = \left(\frac{z_1^n - z_2^n}{z_1 - z_2}\right) - 2\left(\frac{z_2 z_1^n - z_1 z_2^n}{z_1 - z_2}\right).$$

where $z_{1,2} = \frac{1\pm\sqrt{1+4i}}{2}$ are roots of $z^2 - z - i = 0$.

Explicit Summation formula

$$S_n = \frac{1}{1+i-1}\left[C_{n+1} - z_1 z_2 C_n - 1\right].$$

3. **Complex Pell-Lucas sequences** If we take $p = 2,\ q = 1,\ a_1 = 2,\ a_2 = 0,\ b_1 = 2,\ b_2 = 0$ in the equation (1), then we get the Lucas sequence $C_n(C_1, C_2)$ by the following linear complex recurrence relation:

$$C_n(C_1, C_2) = 2C_{n-1} + iC_{n-2}, n \geq 3, \tag{19}$$

with initial terms $C_1 = a_1 + ia_2 = 2$, $C_2 = b_1 + ib_2 = 2$, and $i = \sqrt{-1}$, $a_j, b_j, \in R$, $(j = 1, 2)$ are arbitrary non–zero real numbers.

Generating Function

$$C(z) = \frac{-2z(2+i) - 2i}{z^2 - 2z - i}.$$

Binet Formula

$$C_n = 2\left(\frac{z_1^n - z_2^n}{z_1 - z_2}\right) - 2\left(\frac{z_2 z_1^n - z_1 z_2^n}{z_1 - z_2}\right),$$

where $z_{1,2} = \frac{2\pm\sqrt{2^2+4i}}{2}$ are roots of $z^2 - 2z - i = 0$.

$$S_n = \frac{1}{2+i-1}\left[C_{n+1} - z_1 z_2 C_n\right].$$

4. **Complex Modified Pell sequences**

If we take $p = 2,\ q = 1,\ a_1 = 1,\ a_2 = 0,\ b_1 = 1,\ b_2 = 0$ in the equation (1), then we get the Lucas sequence $C_n(C_1, C_2)$ by the following linear complex recurrence relation:

$$C_n(C_1, C_2) = 2C_{n-1} + iC_{n-2}, n \geq 3, \tag{20}$$

with initial terms $C_1 = a_1 + ia_2 = 1$, $C_2 = b_1 + ib_2 = 1$, and $i = \sqrt{-1}$, $a_j, b_j, \in R$, $(j = 1, 2)$ are arbitrary non–zero real numbers.

Generating Function:

$$C(z) = \frac{-z(2+i) - i}{z^2 - 2z - i}.$$

Binet Formula

$$C_n = \left(\frac{z_1^n - z_2^n}{z_1 - z_2}\right) - \left(\frac{z_2 z_1^n - z_1 z_2^n}{z_1 - z_2}\right)$$

where $z_{1,2} = \frac{2\pm\sqrt{2^2+4i}}{2}$ are roots of $z^2 - 2z - i = 0$.

Explicit Summation formula

$$S_n = \frac{1}{2+i-1}\left[C_{n+1} - z_1 z_2 C_n\right].$$

5. Similarly on restricting the parameters $p = 1,\ q = 2,\ a_1 = 0,\ a_2 = 0,\ b_1 = 1,\ b_2 = 0,\ p = 1,\ q = 2,\ a_1 = 2,\ a_2 = 0,\ b_1 = 1,\ b_2 = 0$ and $p = 2a, q = (b - a^2),\ a_1 = 0,\ a_2 = 0,\ b_1 = 1,\ b_2 = 0$, in the equation (1), we obtain the corresponding results for **Jacobsthal sequences** , **Jacobsthal-Lucas sequences**, **Goksal Bilgici sequences using condition one** [6] ,**Goksal Bilgici sequences using condition second** [6]sequences.

10 Tridiagonal matrix representation

Definition 10.1 We express generalized complex Fibonacci sequence $C_n(p, q, C_1, C_2)$ by the tridiagonal matrix for Generalized Gaussian Fibonacci, whose determinants form the generalized complex Fibonacci sequences as

$$M_n(p,q,C_1,C_2) = \begin{pmatrix} C_2 & -q & 0 & 0 & \cdots & 0 \\ iC_1 & p & -q & 0 & \cdots & 0 \\ 0 & i & p & -q & \cdots & 0 \\ 0 & 0 & i & p & \cdots & 0 \\ \vdots & \vdots & 0 & \vdots & \ddots & -q \\ 0 & 0 & 0 & 0 & i & p \end{pmatrix}_{n\times n} \tag{21}$$

with initial terms $C_1 = a_1 + ia_2$, $C_2 = b_1 + ib_2$, and $i = \sqrt{-1}$, $a_j, b_j, p, q (\neq 0) \in R$ are arbitrary real numbers.

Theorem 10.1. *If $M_n(p,q,C_1,C_2)$ denote the $n \times n, n \geq 1$ tridiagonal matrix for generalized complex Fibonacci sequences as*

$$M_n(p,q,C_1,C_2) = \begin{pmatrix} C_2 & -q & 0 & 0 & \cdots & 0 \\ iC_1 & p & -q & 0 & \cdots & 0 \\ 0 & i & p & -q & \cdots & 0 \\ 0 & 0 & i & p & \cdots & 0 \\ \vdots & \vdots & 0 & \vdots & \ddots & -q \\ 0 & 0 & 0 & 0 & i & p \end{pmatrix}_{n\times n}$$

where C_1 and C_2 are defined above, and $\det M_0(p,q,C_1,C_2) = C_1$.
Then for $n \geq 2$, $C_{n+1} = pC_n + iqC_{n-1}$ $\det M_n(p,q,C_1,C_2) = C_{n+1}(p,q,C_1,C_2)$, *general $(n+1)^{th}$ term of the generalized Gaussian Fibonacci numbers.*

Proof. For we have $n = 1, 2, 3\cdots$, we have

$$\det M_1(p,q,C_1,C_2) = C_1(p,q,C_1,C_2) = C_2,$$

$$\det M_2(p,q,C_1,C_2) = C_2(p,q,C_1,C_2) = pC_2 + iqC_1,$$

$$\det M_3(p,q,C_1,C_2) = C_3(p,q,C_1,C_2) = p^2C_2 + i(pqC_1 + qC_2),$$

Assume that the Theorem is true for $n-1$ and $n-2$, then

$$\det M_{n-1}(p,q,C_1,C_2) = C_n(p,q,C_1,C_2)$$

and

$$\det M_{n-2}(p,q,C_1,C_2) = C_{n-1}(p,q,C_1,C_2)$$

Since $\det M_n = m_{n,n} \det M_{n-1} - m_{n,n-1} m_{n-1,n} C_1 \det M_{n-2}$

$$\Rightarrow \det M_n = p \det M_{n-1} - i(-q) \det M_{n-2}$$

$$\Rightarrow C_{n+1} = pC_n + iqC_{n-1}$$

Hence, the theorem follows by induction. □

11 Matrix representation of generalized Gaussian Fibonacci numbers

Now we introduce the matrix representation of generalized Gaussian Fibonacci numbers $C_n(p,q,C_1,C_2)$ by defining 2×2 matrices Q and N as
$Q=\begin{pmatrix} p & iq \\ 1 & 0 \end{pmatrix}$, and $N=\begin{pmatrix} pC_1+iqC_2 & C_2 \\ C_2 & C_1 \end{pmatrix}$.

Theorem 11.1. *If $C_n(p,q,C_1,C_2)=pC_{n-1}+iqC_{n-2}, n>2$, is the Generalized Gaussian Fibonacci numbers then the matrix representation is $M^{n-1}N=\begin{pmatrix} C_{n+2} & C_n \\ C_n & C_n \end{pmatrix}$.*

Proof. For $n=1$. $\begin{pmatrix} p & iq \\ 1 & 0 \end{pmatrix}^{n-1}\begin{pmatrix} pC_1+iqC_2 & C_2 \\ C_2 & C_1 \end{pmatrix}=\begin{pmatrix} pC_1+iqC_2 & C_2 \\ C_2 & C_1 \end{pmatrix}=\begin{pmatrix} C_3 & C_2 \\ C_2 & C_1 \end{pmatrix}$. Assume that the theorem holds for $n=m$, that is

$$M^{m-1}N=\begin{pmatrix} C_{m+2} & C_{m+1} \\ C_{m+1} & C_m \end{pmatrix}$$

Now for $n=m+1$ we have

$$M^{m}N=\begin{pmatrix} p & iq \\ 1 & 0 \end{pmatrix}^{m}\begin{pmatrix} pC_1+iqC_2 & C_2 \\ C_2 & C_1 \end{pmatrix}$$

$=\begin{pmatrix} p & iq \\ 1 & 0 \end{pmatrix}\begin{pmatrix} p & iq \\ 1 & 0 \end{pmatrix}^{m-1}\begin{pmatrix} pC_1+iqC_2 & C_2 \\ C_2 & C_1 \end{pmatrix}=\begin{pmatrix} p & iq \\ 1 & 0 \end{pmatrix}\begin{pmatrix} C_{m+2} & C_{m+1} \\ C_{m+1} & C_m \end{pmatrix}=\begin{pmatrix} C_{m+3} & C_{m+2} \\ C_{m+2} & C_{m+1} \end{pmatrix}$.
Hence, the theorem follows by induction. □

12 Polar form of Generalized complex Fibonacci numbers

Theorem 12.1. *Considering the polar coordinate for equation (1), we have*

$$r=\left|\frac{p^2}{4}+iq\right|,\ \theta=\arg\left(\frac{p^2}{4}+iq\right),$$

then for $n=0,1$. $\alpha_1,\alpha_2=\frac{p}{2}\pm\sqrt{r}e^{\frac{i(2n\pi+\theta)}{2}}$, then the Generalized complex Fibonacci numbers formula is

$$C_n=\left(\frac{\alpha_1^n-\alpha_2^n}{\alpha_1-\alpha_2}\right)C_2-\left(\frac{\alpha_2\alpha_1^n-\alpha_1\alpha_2^n}{\alpha_1-\alpha_2}\right)C_1.$$

13 Conclusion

This paper introduces a new generalized complex Fibonacci sequence that extends classical recursions into the complex domain. The authors use the Z-transform to derive the sequence's generating function, as well as explicit Binet-type formulas and summation identities for both even and odd terms. This framework is significant because it unifies several well-known sequences, including Fibonacci, Lucas, Pell, and Jacobsthal numbers, along with their Gaussian variants, demonstrating its broad applicability. The work not only advances the theoretical foundation of complex recursive sequences but also suggests future research directions, such as extending the framework to higher-order recursions and exploring potential applications in fields like coding theory and cryptography.

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