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Leonardo Numbers and their Bicomplex Extension

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Abstract: *This paper introduces a new type of Leonardo numbers, referred to as bicomplex Leonardoi numbers. Also, some important relations, including the generating function, Binet's formula, D'Ocagne's identity, Cassini's identity, and Catalan's identity. Furthermore, we present the relationship between Lucas, Fibonacci, and Leonardo numbers.*

Keywords: *Bicomplex Leonardo numbers, Binet's formulas, Fibonacci numbers, Leonardo numbers, Lucas numbers*

1. Introduction and Motivation

The search for natural extensions of complex numbers has long been of interest, with numerous mathematicians exploring such generalizations through the introduction of multicomplex numbers and their associated function theories. Among these extensions, quaternions introduced by S.W. Hamilton [4] and bicomplex numbers introduced by C. Segre [13] for modeling physical phenomena in four-dimensional space are notable. These two systems differ in terms of commutativity; bicomplex numbers are commutative, and quaternions are non-commutative.

Further contributions were made by Price [11], who developed the algebraic structure and function theory of bicomplex numbers. Mathematically, the bicomplex system can be viewed as a two-dimensional Clifford algebra over the complex field, characterized by commutative multiplication. This framework has found significant applications in fields such as image processing, geometry, and theoretical physics [11,12].

A complex number $y \in \mathbb{C}$ can be expressed in the form of

$$y = y_1 + y_2 \mathrm{i}, \text{ such that, } y_1, y_2 \in \mathbb{R}, \; i^2 = -1$$

On the other hand, a bicomplex number $x \in \mathbb{C}_2$is given by

$$x = a_1 + b_1 i + c_1 j + d_1 ij \qquad (1.1)$$

where $a_1, b_1, c_1\ and d_1 \in \mathbb{R},\ and\ i^2 = -1,\ j^2 = -1, (ij)^2 = (ji)^2 = k^2 = 1$.

Also, $ij = k \in \mathbb{C}_2, but\ ij = k \notin \mathbb{C}$. The space bicomplex numbers $\mathbb{C}_2$ has 4 dimensions over $\mathbb{R}$, but the space of complex numbers $\mathbb{C}$ is of dimension 2 over $\mathbb{R}$ [14]. Let the two bicomplex numbers

$$x = a_1 + b_1 i + c_1 j + d_1 ij$$

and

$$y = a_2 + b_2 i + c_2 j + d_2 ij.$$

Then the addition, subtraction and multiplication of bicomplex numbers x and y are given by

$$x \pm y = (a_1 \pm a_2) + (b_1 \pm b_2)i + (c_1 \pm c_2)j + (d_1 \pm d_2)ij \qquad (1.2)$$

And

$$x \times y = (a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2) + (a_1b_2 + b_1a_2 - c_1d_2 - d_1c_2)\, i + (a_1c_2 + c_1a_2 - b_1d_2 - d_1b_2)j + (a_1d_2 + d_1a_2 + b_1c_2 + c_1b_2)ij. \quad (1.3)$$

The multiplication by a real scalar α is given by

$$\alpha x = \alpha a_1 + \alpha b_1 i + \alpha c_1 j + \alpha d_1 ij \quad (1.4)$$

Numerous studies in the literature explore Fibonacci and Lucas numbers [5,6,8]. These sequences have also been investigated within various number system, including quaternions and hybrid numbers [3,7,10,15].

In this work, we present a brief overview of the Fibonacci and Lucas sequences.

The Fibonacci numbers introduced by the Italian mathematician Leonardo Fibonacci are defined by the recurrence relation in which each term is obtained by adding the two preceding terms.

The sequence starts as:

0,1,1,2,3,5,8,13,21,34,55,89,144,233,377,610,987,1597,2584, 4181,6765,

10496, 17711,28657,46368, 75025, ...

Mathematically, the Fibonacci sequence is represented by F_n and follows the linear recurrence relation:

$$F_n = F_{n-1} + F_{n-2}\,, \text{ for } n \geq 2, \quad (1.5)$$

$$F_{n+2} = F_{n+1} + F_n \text{ for } n \geq 0, \text{with } F_0 = 0\,, F_1 = F_2 = 1.$$

On the other hand, Lucas numbers were named by the French mathematician Édouard Lucas, who formalized this sequence in the 19th century. Lucas is often credited as the "father" of these numbers due to his extensive work on both the Fibonacci and Lucas sequences.[5]

The Lucas sequence follows the same recurrence relation as the Fibonacci sequence but starts with different initial terms.

It is defined as:

2,1,3,4,7,11,18,29,47,76,123,199,322,521,843,1364,2207,3571,5778,9349,15127,

24476,39603, 64079,103682,167761, ...

The general recurrence formula is

$$L_n = L_{n-1} + L_{n-2} \text{ for } n \geq 2 \quad (1.6)$$

$$L_{n+2} = L_{n+1} + L_n \text{ for } n \geq 0\,, \text{with } L_0 = 2, L_1 = 1.$$

For $n \geq 0$, the Binet representations of Fibonacci number F_n and the Lucas number L_n are expressed as

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad L_n = \alpha^n + \beta^n$$

where α and β are the roots of

$$y^2 - y - 1 = 0, \text{i.e. } \alpha = \frac{(1+\sqrt{5})}{2} \text{ and } \beta = \frac{(1-\sqrt{5})}{2}.$$

There are many relations between Fibonacci and Lucas sequences. Which are as follows:

Lemma 1.1 For $n \geq 1$, the relation holds,[2].

a. $F_n = \frac{L_{n-1}+L_{n+1}}{5}$, (1.7)

b. $F_{n+2} = \frac{L_{n-1}+L_{n+5}}{10}$, (1.8)

c. $L_n = F_n + 2F_{n-1}$, (1.9)

In this paper, we express the Leonardo sequence, which has similar properties to the Fibonacci and Lucas sequence, and denote the n^{th} Leonardo number byL_n^e. We present the Leonardo sequence along with the statement of the respective Binet's formula, sum and product formulas, several identities, and the generating function.

2. Leonardo Number

Leonardo's number was introduced by Catrino and Borgs [2], and is denoted by L_n^e, and the Leonardo sequence is given by the recurrence relation

$$L_n^e = L_{n-1}^e + L_{n-2}^e + 1 \text{ for } n \geq 2 \quad (2.1)$$

With initial condition:

$$L_0^e = L_1^e = 1$$

The Leonardo numbers are
1,1,3,5,9,15,25,41,67,109,177,287,465,753,1219,1973,3193,5167,

8361,13529,21891,35421,57323,92745,...

Also, the relations hold for Leonardo numbers for $n \geq 2$,

$$L_{n+1}^e = 2L_n^e - L_{n-2}^e \quad (2.2)$$

Lemma 2.1. For $n \geq 2$, L_n^e is an odd number.

Lemma 2.2. For $n \geq 0$, $L_n^e = 2F_{n+1} - 1$.

Proof:

We prove the property of Leonardo numbers by induction on *n*. For $n = 0$ and $n = 1$, it is verified.

Now we assume that it is true $for\ n = k, i.e. L_k^e = 2F_{k+1} - 1$.

Now we show that it is true $n = k + 1$.

$$L_{k+1}^e = L_k^e + L_{k-1}^e + 1$$

$$= (2F_{k+1} - 1) + (2F_k - 1) + 1$$

$$= 2(F_{k+1} + F_k) - 1$$

$$L_{k+1}^e = 2F_{k+2} - 1.$$

which proves for $n = k + 1$ so it is true for $n \geq 0$.

Thus the result is verified .

□

Lemma 2.3. a) For $n \geq 0$, $L_n^e = L_{n+2} - F_{n+2} - 1$,

b) For $n \geq 0$, $L_n^e = \frac{2(L_n+L_{n+2})}{5} - 1$,

c) For $n \geq 0$, $L_{n+3}^e = \frac{(L_{n+1}+L_{n+7})}{5} - 1$, where L_n^e is the n^{th} Leonardo number and L_n is the n^{th} Lucas number and F_n is the n^{th} Fibonacci number.

Lemma 2.4.[6] (Binet's formula) For $n \geq 0, L_n^e = 2\left(\frac{w^{n+1} - z^{n+1}}{w-z}\right) - 1,$
where L_n^e is the bicomplex Leonardo number and
w and z are the roots of $y^3 - 2y^2 + 1 = 0, i.e. w = \frac{(1+\sqrt{5})}{2}, z = \frac{(1-\sqrt{5})}{2}$.

Catarino and Borges [2] derived Cassini's, Catalan's, and d'Ocagne's identities for Leonardo numbers, along with establishing their two-dimensional recurrence relations and matrix form. Shannon [14] later introduced generalized Leonardo numbers, which encompass Asveld's extension and Horadam's [6] generalized sequence. Below, we revisit several identities associated with Fibonacci, Lucas, and Leonardo numbers. Further details can be found in [2,8,15,17].

$$F_n + L_n = 2F_{n+1} \qquad (2.3)$$

$$F_{n+r} + F_{n+s} - F_n F_{n+r+s} = (-1)^n F_r F_s \qquad (2.4)$$

$$L_{n+m}^e + (-1)^m L_{n-m}^e = L_m(L_n^e + 1) - 1 - (-1)^m \qquad (2.5)$$

$$L_{n+m}^e - (-1)^m L_{n-m}^e = L_{n+1}(L_{m-1}^e + 1) - 1 + (-1)^m \qquad (2.6)$$

$$F_n^2 - F_{n+r}F_{n-r} = (-1)^{n-r} F_r^2 \qquad (2.7)$$

$$L_{s+r} - (-1)^s L_{r-s} = 5F_r F_s \qquad (2.8)$$

$$\sum_{k=0}^{n} (-1)^{k-1} F_{k+1} = (-1)^{n-1} F_n \qquad (2.9)$$

Various extensions of Fibonacci and Lucas numbers have been explored in the literature. Among them, Nurkan et al. [10] introduced the bicomplex Fibonacci and Lucas numbers and investigated several of their properties. The bicomplex forms of these sequences are defined as follows:

$$BF_k = F_k + F_{k+1}i + F_{k+2}j + F_{k+3}\, ij$$

and

$$BL_k = L_k + L_{k+1}i + L_{k+2}j + L_{k+3}\, ij.$$

where $F_k and\ L_k$ are k^{th} Fibonacci and Lucas sequences, respectively, and

$$i^2 = j^2 = -1, \ ij = ji, (ij)^2 = k^2 = 1\,.$$

Motivated by the above-mentioned works, we propose the concept of bicomplex numbers constructed from Leonardo number components. Our objective is to derive the generating function, Binet's formula, recurrence relations, summation expressions, and identities such as those of Cassini, Catalan, along with other related results.

3. Bicomplex Leonardo Number

Now we introduce Bicomplex Leonardo number , Binet's formula, along with Catalan's and Cassini's identities, and summation formula, and generating function.

Definition 3.1. The r^{th} bicomplex Leonardo number is denoted by BL_r^e and is defined by

$$BL_r^e = L_r^e + iL_{r+1}^e + j\, L_{r+2}^e + kL_{r+3}^e\ . \quad \text{for } n \geq 1 \qquad (3.1)$$

From above relation of Leonardo number and definition of bicomplex Leonardo number we conclude that

$$BL_r^e = L_r^e + iL_{r+1}^e + j\,L_{r+2}^e + k\,L_{r+3}^e$$

$$BL_r^e = (L_{r-1}^e + L_{r-2}^e + 1) + i(L_r^e + L_{r-1}^e + 1) + j(L_{r+1}^e + L_r^e + 1) + k(L_{r+2}^e + L_{r+1}^e + 1)$$

$$= [(L_{r-1}^e + iL_r^e + jL_{r+1}^e + kL_{r+2}^e) + (L_{r-2}^e + iL_{r-1}^e + jL_r^e + kL_{r+1}^e) + (1 + i + j + k)]\ BL_r^e$$

$$= BL_{r-1}^e + BL_{r-2}^e + P \qquad (3.2)$$

where $P = (1 + i + j + k)$.

For starting case of $BL_r^e = L_r^e + iL_{r+1}^e + j\,L_{r+2}^e + k\,L_{r+3}^e$

If we take the value of $r = 0,1,2,3, \ldots$, then we have

$BL_0^e = 1 + i + 3j + 5k,$

$BL_1^e = 1 + 3i + 5j + 9k,$

$BL_2^e = 3 + 5i + 9j + 15k,$

$BL_3^e = 5 + 9i + 15j + 25k,$ and so on.

Using the above relation, we can find another recurrence relation of bicomplex Leonardo numbers, i.e., $BL_{r+1}^e = L_{r+1}^e + iL_{r+2}^e + j\,L_{r+3}^e + kL_{r+4}^e$

$$= (2L_r^e - L_{r-2}^e) + i(2L_{r+1}^e - L_{r-1}^e) + j(2L_{r+2}^e - L_r^e) + k(2L_{r+3}^e - L_{r+1}^e)$$

$$= 2(L_r^e + iL_{r+1}^e + jL_{r+2}^e + kL_{r+3}^e) - (L_{r-2}^e + iL_{r-1}^e + jL_r^e + kL_{r+1}^e)$$

$$BL_{r+1}^e = 2BL_r^e - BL_{r-2}^e \qquad (3.3)$$

Operations on bicomplex Leonardo numbers

The addition, subtraction and product of two bicomplex Leonardo numbers are as follows:

Let $BL_r^e\ and\ BL_s^e$ be two bicomplex Leonardo number then

$$BL_r^e = L_r^e + iL_{r+1}^e + jL_{r+2}^e + kL_{r+3}^e \qquad (3.4)$$

$$BL_s^e = L_s^e + iL_{s+1}^e + jL_{s+2}^e + kL_{s+3}^e \qquad (3.5)$$

The sum and difference of two bicomplex Leonardo are as follows:

$$BL_r^e \pm BL_s^e = (L_r^e + iL_{r+1}^e + jL_{r+2}^e + kL_{r+3}^e) \pm (L_s^e + iL_{s+1}^e + jL_{s+2}^e + kL_{s+3}^e)$$

$$= (L_r^e \pm L_s^e) + i(L_{r+1}^e \pm L_{s+1}^e) + j(L_{r+2}^e \pm L_{s+2}^e) + k(L_{r+3}^e \pm L_{s+3}^e) \qquad (3.6)$$

The multiplication table of $i, j, and\ k(= ij)$ is given below

·	**1**	***i***	***j***	***k*(=ij)**
1	1	i	j	k
i	i	-1	k	-j
j	j	k	-1	-i
***k*(=*ij*)**	k	-j	-i	1

The multiplication of bicomplex Leonardo numbers are

$$BL_r^e \cdot BL_s^e = (L_r^e + iL_{r+1}^e + jL_{r+2}^e + kL_{r+3}^e) \cdot (L_s^e + iL_{s+1}^e + jL_{s+2}^e + kL_{s+3}^e)$$

$$= [(L_r^e L_s^e - L_{r+1}^e L_{s+1}^e - L_{r+2}^e L_{s+2}^e + L_{r+3}^e L_{s+3}^e)$$

$$+i(L_r^e L_{s+1}^e + L_{r+1}^e L_s^e - L_{r+2}^e L_{s+3}^e - L_{r+3}^e L_{s+2}^e)$$

$$+j(L_r^e L_{s+2}^e + L_{r+2}^e L_s^e - L_{r+1}^e L_{s+3}^e - L_{r+3}^e L_{s+1}^e)$$

$$+k((L_r^e L_{s+3}^e + L_{r+3}^e L_s^e + L_{r+1}^e L_{s+2}^e + L_{r+2}^e L_{s+1}^e)]$$

$$= BL_s^e \cdot BL_r^e$$

The conjugate of bicomplex Leonardo numbers [12] with *i,j,* and *k* units is

$$(BL_r^e)^i = L_r^e - iL_{r+1}^e + jL_{r+2}^e - kL_{r+3}^e$$

$$(BL_r^e)^j = L_r^e + iL_{r+1}^e - jL_{r+2}^e - kL_{r+3}^e$$

$$(BL_r^e)^k = L_r^e - iL_{r+1}^e - jL_{r+2}^e + kL_{r+3}^e$$

Bicomplex Leonardo numbers are an extension of the classical Leonardo numbers into the bicomplex number system. The following are the related theorems.

Theorem 3.1. If $(BL_r^e)^i, (BL_r^e)^j and\ (BL_r^e)^k$, are conjugation of bicomplex Leonardo numbers, then

a. $BL_r^e \cdot (BL_r^e)^i = {L_r^e}^2 + {L_{r+1}^e}^2 - {L_{r+2}^e}^2 - {L_{r+3}^e}^2 + 2j\, L_{r+1}^e L_{r+3}^e$
b. $BL_r^e \cdot (BL_r^e)^j = {L_r^e}^2 - {L_{r+1}^e}^2 + {L_{r+2}^e}^2 - {L_{r+3}^e}^2 + 2i(L_r^e L_{r+1}^e + L_{r+2}^e L_{r+3}^e)$
c. $BL_r^e \cdot (BL_r^e)^k = {L_r^e}^2 + {L_{r+1}^e}^2 + {L_{r+2}^e}^2 + {L_{r+3}^e}^2 + 2ij(L_r^e L_{r+3}^e - L_{r+1}^e L_{r+2}^e)$

Theorem 3.2. If BL_r^e be the r^{th} bicomplex Leonardo number, then for any integer $r \geq 0$,

$$BL_r^e = 2BF_{r+1} - P$$

where BF_r is the r^{th} bicomplex Fibonacci number and $P = 1 + i + j + k$.

Proof:

By using the recurrence relation of Leonardo and Fibonacci and the definition of a bicomplex Leonardo number, we have,

$$BL_r^e = L_r^e + iL_{r+1}^e + jL_{r+2}^e + kL_{r+3}^e$$

$$= (2F_{r+1} - 1) + 2(F_{r+2} - 1)i \quad +2(F_{r+3} - 1)j + 2(F_{r+4} - 1)k$$

$$= 2(F_{r+1} + F_{r+2}i + F_{r+3}j + F_{r+4}k\ ) - (1 + i + j + k)$$

$$= 2BF_{r+1} - P$$ □

Theorem 3.3. If BL_r^e be the r^{th} bicomplex Leonardo number, then for $r \geq 0$,

$$BL_r^e = BF_r + BL_r - P$$

where $P = 1 + i + j + k$

and BF_r and BL_r are the r^{th} bicomplex Fibonacci and bicomplex Lucas numbers.

Proof:

By using the definition of a Bicomplex Leonardo number, we have

$$BL_r^e = L_r^e + iL_{r+1}^e + jL_{r+2}^e + k$$

Which can be expressed as with the help of (Lemma 2.2 and equation (2.3)

$$= (2F_{r+1} - 1) + i(2F_{r+2} - 1) + j(2F_{r+3} - 1) + k(2F_{r+4} - 1)$$

$$= (F_r + L_r - 1) + i(F_{r+1} + L_{r+1} - 1) + j(F_{r+2} + L_{r+2} - 1)$$
$$+k(F_{r+3} + L_{r+3} - 1)$$

$$= (F_r + iF_{r+1} + jF_{r+2} + kF_{r+3}) + (L_r + iL_{r+1} + jL_{r+2} + kL_{r+3}) - (1 + i + j + k)$$

$$= BF_r + BL_r - P.$$

Theorem 3.4. The generating function corresponding to the bicomplex Leonardo numbers, represented by $gBL_r^e(t)$ is given by

$$gBL_r^e(t) = \sum_{r=0}^{\infty} BL_r^e\, t^r = \frac{BL_0^e + t(-1+i-j-k) + t^2(1-i-j-3k)}{1-2t+t^3}$$

Proof:

The power series of the generating function is

$$gBL_r^e(t) = \sum_{r=0}^{\infty} BL_r^e t^r = BL_0^e + tBL_1^e + t^2 BL_2^e + \cdots + t^r BL_r^e + \cdots$$

Then by multiplying both sides by $(1 - 2t + t^3)$, we have

$$(1 - 2t + t^3) gBL_r^e(t) = (1 - 2t + t^3)(BL_0^e + tBL_1^e + t^2 BL_2^e + \cdots + t^r BL_r^e + \cdots)$$

$$= (BL_0^e + tBL_1^e + t^2 BL_2^e + t^3 BL_3^e \ldots) + (-2tBL_0^e - 2t^2 BL_1^e - 2t^3 BL_2^e - \cdots)$$
$$+(t^3 BL_0^e + t^4 BL_1^e + t^5 BL_2^e + \cdots) + \cdots$$

Arranging the terms, we get

$$= BL_0^e + t(BL_1^e - 2BL_0^e) + t^2(BL_2^e - 2BL_1^e) + t^3(BL_3^e - 2BL_2^e + BL_0^e) + \cdots$$

$$= BL_0^e + t(-1 + i - j - k) + t^2(1 - i - j - 3k)$$

where $(BL_1^e - 2BL_0^e) = (-1 + i - j - k)$

$(BL_2^e - 2BL_1^e) = (1 - i - j - 3k)$

$(BL_3^e - 2BL_2^e + BL_0^e) = 0$

In this way, the coefficient of $t^4, t^5, \ldots$ are zero.

Hence $(1 - 2t + t^3) gBL_r^e(t) = BL_0^e + t(-1 + i - j - k) + t^2(1 - i - j - 3k)$

$$gBL_r^e(t) = \frac{BL_0^e + t(-1 + i - j - k) + t^2(1 - i - j - 3k)}{(1 - 2t + t^3)}$$

□

Theorem 3.5. The Binet's formula for a bicomplex Leonardo number is as follows:

$$\text{For } r \geq 0, \qquad BL_r^e = \frac{2\bar{w}\, w^{n+1} - 2\bar{z}\, z^{n+1}}{w - z} - P$$

where

$\bar{w} = 1 + iw + jw^2 + kw^3$, $w = \frac{1+\sqrt{5}}{2}$

$\bar{z} = 1 + iz + jz^2 + kz^3$, $z = \frac{1-\sqrt{5}}{2}$

$P = 1 + i + j + k$

Proof:

Based on the definition of the bicomplex Leonardo numbers and the Binet's formula for Leonardo numbers [10], it follows that

$$BL_r^e = L_r^e + iL_{r+1}^e + j\, L_{r+2}^e + kL_{r+3}^e$$

and with the help of

$$L_r^e = \frac{2w^{r+1} - 2z^{r+1}}{w - z} - 1$$

$$BL_r^e = \left(\frac{2w^{r+1} - 2z^{r+1}}{w-z} - 1\right) + i\left(\frac{2w^{r+2} - 2z^{r+2}}{w-z} - 1\right) + j\left(\frac{2w^{r+3} - 2z^{r+3}}{w-z} - 1\right) + k\left(\frac{2w^{r+4} - 2z^{r+4}}{w-z} - 1\right).$$

Here if we use

$$\bar{w} = 1 + iw + jw^2 + kw^3 \text{ and } \bar{z} = 1 + iz + jz^2 + kz^3$$

Thus we obtain the result

$$BL_r^e = \frac{2\bar{w}\, w^{n+1} - 2\bar{z}\, z^{n+1}}{w - z} - P.$$

□

Theorem 3.6. (D'Ocagne's identity)

Let BL_r^e be the bicomplex Leonardo number, for $r \geq s + 1$, the following relation is valid.

$$BL_r^e \cdot BL_{s+1}^e - BL_{r+1}^e \cdot BL_s^e = 12(-1)^{s+1} F_{r-s}(2j + k) - 2(1 + i + j + k)(+BF_s - BF_r).$$

Proof:

By using the relation, $BL_r^e = 2BF_{r+1} - P$,

where $P = 1 + i + j + k$ and D'Ocagne's identity of bicomplex Fibonacci number [10]

$$\begin{aligned} BL_r^e \cdot BL_{s+1}^e - BL_{r+1}^e \cdot BL_s^e &= (2BF_{r+1} - P)(2BF_{s+2} - P) - (2BF_{r+2} - P)(2BF_{s+1} - P) \\ &= 4(BF_{r+1}BF_{s+2} - BF_{r+2}BF_{s+1}) - 2P(BF_{r+1} + BF_{s+2} - BF_{r+2} - BF_{s+1}) \\ &= 4[3(-1)^{s+1} F_{r-s}(2j + k)] - 2P(BF_s - BF_r) \\ &= 12(-1)^{s+1} F_{r-s}\, (2j + k) - 2(1 + i + j + k)(BF_s - BF_r). \end{aligned}$$

Here by use the relation $F_m F_{n+1} - F_n F_{m+1} = (-1)^n F_{m-n}$ [8,15] □

Theorem3.7. Cassini's Identity

Let BL_r^e be the bicomplex Leonardo number. For $r \geq 1$, the equality holds:

$BL_{r-1}^e BL_{r+1}^e - {BL_r^e}^2 = 12\big((-1)^{r+1}(2j + k)\big) - 2P(BF_{r-2})$, where $P = 1 + i + j + k$

Proof:

We have $BL_r^e = 2BF_{r+1} - P$ and the relation of Fibonacci number [8,15]

$$F_r F_{s+1} - F_{r+1} F_s = (-1)^s F_{r-s}$$

Then we have

$$BL_{r-1}^{e}BL_{r+1}^{e} - {BL_{r}^{e}}^{2} = (2BF_r - P)(2BF_{r+2} - P) - (2BF_{r+1} - P)^2$$

$$= 4(BF_r\, BF_{r+2} - BF_{r+1}BF_{r+1}) + 2(-BF_r + BF_{r+1})P + 2P(-BF_{r+2} + BF_{r+1})$$

$$= 4(3(-1)^{r+1}(2j + k) + 2PBF_{r-1} + 2PBF_r$$

$$= 12(-1)^{r+1}(2j + k) - 2P(BF_{r-1} + BF_r)$$

$$= 12(-1)^{r+1}(2j + k) - 2P(BF_{r-2}).$$

Theorem:3.8 Catalan's identity

Let BL_n^e be the bicomplex Leonardo number. For, $n \geq 1$, the equality holds:

$${BL_n^e}^2 - BL_{n-r}^e BL_{n+r}^e = 4[3(-1)^{n-r+1}F_r^2(2j + k)] - 2P(2BF_{n+1} - BF_{n-r+1} - BF_{n+r+1})$$

Proof:

By using the relation $BL_n^e = 2BF_{n+1} - P$, $L_n^e = 2F_{n+1} - 1$ and Catalan's identity of bicomplex Fibonacci numbers [1, 10].

The n^{th} bicomplex Leonardo number is given by

$$BL_n^e = L_n^e + iL_{n+1}^e + jL_{n+2}^e + kL_{n+3}^e \text{ and } P = 1 + i + j + k$$

$$= (2F_{n+1} - 1) + i(2F_{n+2} - 1) + j(2F_{n+3} - 1) + k(2F_{n+4} - 1)$$

$$= 2BF_{n+1} - P$$

Now we have

$${BL_n^e}^2 - BL_{n-r}^e BL_{n+r}^e = (2BF_{n+1} - P)^2 - (2BF_{n-r+1} - P)(2BF_{n+r+1} - P)$$

$$= 4(BF_{n+1}^2 - BF_{n-r+1}\, BF_{n+r+1}) - 2P(2BF_{n+1} - BF_{n-r+1} - BF_{n+r+1})$$

In view of the relation

$$F_m^2 - F_{m-r}F_{m+r} = (-1)^{m-r}F_r^2$$

and

$$BF_{n+1}^2 - BF_{n-r+1}\, BF_{n+r+1} = 3\,(-1)^{n-r+1}F_r^2(2j + k)$$

We have

$${BL_n^e}^2 - BL_{n-r}^e BL_{n+r}^e = 4[3\,(-1)^{n-r+1}F_r^2(2j + k)] - 2P(2BF_{n+1} - BF_{n-r+1} - BF_{n+r+1}).$$

Conclusion

In this research article, we present the definition of bicomplex Leonardo numbers, formulated using the coefficients from the Leonardo sequence. Furthermore, by exploring their connection with bicomplex Fibonacci numbers, we have derived D'Ocagne's, Catalan's and Cassini's identities.

Author Contribution Statements

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Declaration of Competing Interests

The corresponding author, on behalf of all authors, confirms that there are no competing interests.

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