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TRANSSHIPMENT CONTRAFLOW ON MULTI-TERMINAL NETWORKS

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Abstract: Contraflow technique is the widely accepted model on network optimization. It allows arc reversal that increases the arc capacities. The earliest arrival transshipment contraflow is an important model that transship the given flow value by sending the maximum amount at each time point from the beginning within given time period by reversing the direction arcs from the sources to the sinks at time zero. This problem has not been solved polynomially on complex networks, i.e., multi-terminal networks yet. However, its 2-value-approximation solution has been found by Pyakurel and Dhamala [13] in pseudo-polynomial time complexity. Moreover, they have claimed that for the special case of zero transit time on each arc, the 2-value-approximation solution can be computed in polynomial time complexity. In this paper, we solve their claim presenting an efficient algorithm.

Key Words: Network optimization, value-approximation, contraflow, complexity.

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1. Introduction

Contraflow increases the outbound capacities of arcs with the capabillity of arc reversals in the required direction. The obtained network with increased arcs capacities is the auxiliary network. On auxiliary network, contraflow problem not only maximizes the flow value but also minimizes the time to transship the given flow value. From the practice in evacuation planning, the evacuation time is reduced at least 40 percent with at most 30 percent of the total arc reversals, [8]. For the various mathematical models, heuristics, optimization and simulation techniques with contraflow configuration, we refer to Dhamala [1].

From the analytical point of view, we can find that the flow values obtained by contraflow models increase significantly that may be doubled for given time horizon. Moreover, the contraflow model is two times faster than the models without contraflow to transship the given flow value. Some contraflow problems with efficient solution algorithms in particular networks have been solved in [2, 9, 10, 11, 12, 13].

Moreover, using the natural transformation of [4], we have computed some contraflow solutions in different particular network by reversing the direction of arcs at time zero in continuous time model with the same complexity as in discrete time model [15, 3].

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The earliest arrival contraflow problem maximizes flow at every point of time from the beginning with arc reversal capability. If the supplies/demands are given and we have to transship given supplies to satisfied the demands within fixed time horizon, then the problem turns into the earliest arrival transshipment contraflow problem. So far, to the best of the author's knowledge, the earliest arrival transshipment contraflow problem on multi-terminal networks has been studied by Pyakurel and Dhamala [13]. They have presented an pseudo-polynomial time approximation algorithm to solve the problem for arbitrary transit time on each arc of the network. Moreover, they have claimed that an approximate earliest arrival transshipment contraflow problem can be solved on multi-terminal networks with zero transit time on each arc in polynomial time complexity. In zero transit time, arc capacities of networks restrict the quantity of flow that can be sent at any one time with arc reversal capability. In this paper, we solve their claim in detail. We present a polynomial time approximation algorithm to solve the problem accordingly.

The organization of the paper is as follows. In Section 2, we model the earliest arrival transshipment contraflow problem with a short description of required concepts and denotations. In section 3, we present our main results on the earliest arrival transshipment contraflow problem on multi-terminal networks. Section 4 concludes the paper.

2. Preliminaries

Let $G = (V, A)$ be a directed graph with a finite set of nodes V and a finite set of arcs A. We assume that $|V| = n$ and $|A| = m$. As the case is of contraflow, two way network configuration is allowed. Let $S \subset V$ and $D \subset V$ be a set of source nodes which are the starting points of flow and a set of sink nodes with enough capacity, i.e., the final destination of flow. Nodes s and d represent a single source and a single sink.

The network consists of nonnegative functions of arc capacities $b_A : A \to \mathcal{Z}^+$, node capacities $b_V : V \to \mathcal{Z}^+$ and arc transit times $\tau : A \to \mathcal{Z}^+$. The arc capacities $b_A(e),\ e \in A$ represent the maximum units of evacuees that may enter the initial node of arc e per time period. The node capacities $b_V(v),\ v \in V$ bound the amount of evacuees allowed to hold at node v. The time needed to travel one unit of evacuees on the arc $e = (v, w)$ from node v to node w is the transit time $\tau(e)$. The vectors $\mu(s)$ and $\nu(d)$ represent the given supply and demand at each source and sink, respectively. We assume that $A_d^{out} = A_s^{in} = \emptyset$, where $A_v^{out} = \{(v, w) \in A\}$ and $A_v^{in} = \{(w, v) \in A\}$ for the node $v \in V$.

The transportation network $\mathcal{N} = (V, A, b_A, \tau, S, D, \mu(s), \nu(d), T)$ is represented by the collection of all data in the evacuation scenario with predetermine time T. We assume a finite time horizon T that means everything must happen before time T. Time can increase in discrete increments or continuously. We consider the discrete time with a suitable time unit like at times $t = 0, 1, \ldots, T$ and all time related parameters are integers. The choice of time unit effects the problem directly i.e., if the time unit is shorter then the problem is more complex. Let $\mathbf{T}$ be the domain of time i.e., $\mathbf{T} = \{0, 1, \ldots, T\}$.

Let the reversal of an arc $e = (v, w)$ be $e^{-1} = (w, v)$. For a contraflow configuration of a network $\mathcal{N}$ with symmetric travel times, the auxiliary network $\overline{\mathcal{N}} = (V, E, b_E, b_V, \tau, S, D, T)$

consists of the modified arc capacities and travel times as

$$b_E(\overline{e}) = b_A(e) + b_A(e^{-1}) \text{ , and } \tau(\overline{e}) = \begin{cases} \tau(e) & \text{if } e \in A \\ \tau(e^{-1}) & \text{otherwise} \end{cases}$$

where, an edge $\overline{e} \in E$ in $\overline{\mathcal{N}}$ if $e \vee e^{-1} \in A$ in $\mathcal{N}$. The remaining graph structure and data are unaltered.

Let a non-negative function $x_s : A \to \mathcal{R}^+$ represents the static flow and let the value of a static s-d flow x_s be $val(x_s)$. Similarly, let the non-negative function $x_d : A \times \mathbf{T} \to \mathcal{R}^+$ represents the dynamic flow and its value is $val(x_d)$.

Let $\mathcal{N}^R_{x_s} = (V, \overrightarrow{A} \cup \overleftarrow{A})$ be the residual network of $\mathcal{N}$ where $\overrightarrow{A} = \{\overrightarrow{e} = e \mid x_s(e) < b_A(e)\}$ with capacity $b_A(e) - x_s(e)$ and transit time $\tau(e)$, and $\overleftarrow{A} = \{\overleftarrow{e} = (\text{head}(e), \text{tail}(e)) \mid x_s(e) > 0\}$ with capacity $x_s(e)$ and a transit time $-\tau(e)$.

A dynamic s-d flow x_d for given time T with arc reversal capability satisfies the flow conservation and capacity constraints (2.1-2.3). The inequality flow conservation constraints allow to wait flow at intermediate nodes, however, the equality flow conservation constraints force that flow entering an intermediate node must leave it again immediately.

$$\sum_{\sigma=\tau(e)}^{T} \sum_{e \in A_v^{in}} x_d(e, \sigma - \tau(e)) - \sum_{\sigma=0}^{T} \sum_{e \in A_v^{out}} x_d(e, \sigma) \quad = \quad 0, \ \forall \ v \notin \{s, d\} \tag{2.1}$$

$$\sum_{\sigma=\tau(e)}^{t} \sum_{e \in A_v^{in}} x_d(e, \sigma - \tau(e)) - \sum_{\sigma=0}^{t} \sum_{e \in A_v^{out}} x_d(e, \sigma) \quad \geq \quad 0, \forall v \notin \{s, d\}, t \in \mathbf{T} \tag{2.2}$$

$$0 \ \leq \ x_d(e, t) \ \leq \ b_A(e, t), \qquad \forall \ e \in A, \ t \in \mathbf{T} \tag{2.3}$$

The earliest arrival flow (EAF) problem with arc reversal capability maximizes the $val(x_d, t)$ in (2.4) for all $t \in \mathbf{T}$ satisfying the constraints (2.1-2.3). We denote the maximum flow value by $val_{max}(x_d, t)$.

$$val(x_d, t) = \sum_{\sigma=0}^{t} \sum_{e \in A_s^{out}} x_d(e, \sigma) = \sum_{\sigma=\tau(e)}^{t} \sum_{e \in A_d^{in}} x_d(e, \sigma - \tau(e)) \tag{2.4}$$

If the supplies $\mu(s)$ and demands $\nu(d)$ are given and we have to transship given supplies within time T, then the EAF is turns into the earliest arrival transshipment (EAT) problem. For contraflow network, the problem is the earliest arrival transshipment contraflow (EATCF).

For a network $\mathcal{N}$, the time expanded network $\mathcal{N}(T) = (V_T, A_M \cup A_H)$ is defined is defined by copying the network for each time step as presented for static network in [5] as follows:

$$\begin{aligned} V_T &= \{v(t) | v \in V, t \in \{0, 1, ..., T\}\} \text{ .} \\ A_M &= \{e(t) = (v(t), w(t + \tau_e) | e = (v, w) \in A, \theta \in \{0, 1, ..., T - \tau_e\}\} \text{ , for movement arcs.} \\ A_H &= \{(v(t), v(t + 1)) | v \in V, t \in \{0, 1, ..., T - 1\}\} \text{ , for holdover arcs.} \end{aligned}$$

Let $\mathcal{N}^\star$ be the two-terminal extended network of multi-terminal network $\mathcal{N}$ obtained by adding a super-terminal node $(\star)$ and introducing arcs $(\star, s_i)$ to each $s_i \in S$ with infinite capacity and zero transit time, and arcs $(d_i, \star)$ to each $d_i \in D$ with infinite capacity and transit time -$(T+1)$ for given time period T.

Authors in [6, 7] introduced a $\beta-$value-approximate EAT that is a dynamic flow x_d that achieves at every point in time $t \in \{1, ..., T\}$ and at least a $\beta-$ fraction of the maximum flow value can be sent at time t. They proved that proved that $\beta = 2$ is the best possible approximation factor. They presented a 2-value-approximate algorithm that gives 2-value approximate EAT in pseudo-polynomial time complexity. It works on time expanded network $\mathcal{N}(t)$. For each source $s \in S$, there is a source s_0 with t arcs (s_0, s_i) for the t copies of s. Similarly, for each sink $d \in D$, there is a sink d_0 with t arcs (d_i, d_0). Let s^* be a super-source with $|S|$ arcs (s^*, s_0) having capacity $b_A(s_0)$, and d^* a super-sink with $|D|$ edges (d_0, d^*) having capacity $-b_A(d_0)$. Then, maximum flow with time horizon t is equivalent to a maximum static flow from s^* to d^* in $\mathcal{N}(t)$.

For the special networks with zero transit time, authors in [6, 7] has presented Algorithm 2.1 that computes a maximum flow. All the arcs are of the form $e = (v(t), w(t))$ for some time t. The maximum flow is independently computed from previously computed flow in earlier time layers. The same static flow can be sent repeatedly in each time using original network until the supplies at sources shift into sinks. This algorithm has polynomial time complexity.

Algorithm 2.1. *Zero time 2-value-approximate EAT algorithm*

Input: *Given a dynamic network* $\mathcal{N} = (V, A, b_A, \tau, S, D, \nu)$ *with zero transit times* $\tau(e) = 0$.

(1) *Define the supplies* $\nu' = \nu$ *and set* $t = 1$.
(2) *Respecting the supplies* ν', *a maximum static transshipment* x_{stat_t} *is obtained.*
(3) *Let the maximum time needed to transship* x_{stat_t} *flow be* $a(t)$ *until a source or a sink becomes empty:*

$$a(t) = min\left\{\left\lfloor\frac{\nu'(s)}{val(x_{stat_t})}\right\rfloor \mid s \in S, \nu'(s) > 0\right\} \cup \left\{\left\lfloor\frac{-\nu'(d)}{val(x_{stat_t})}\right\rfloor \mid d \in D, \nu'(d) < 0\right\}$$

(4) *Depending of the sending flow* x_{stat_t}, *update the supplies:*
$\nu'(s) = \nu'(s) - a(t).ex(x_{stat_t})$ *for* $s \in S$ *with* $\nu'(s) > 0$,
$\nu'(d) = \nu'(d) + a(t).ex(x_{stat_t})$ *for* $d \in D$ *with* $\nu'(d) < 0$.
(5) *If* $\nu' \neq 0$, *set* $t = t + 1$ *and continue with Step 2.*

Output: *The dynamic flow that sends* x_{stat_t} *units starting at* $\sum_{j=1}^{t-1} a(t)$ *for* $a(t)$ *time units.*

Theorem 2.2. [6, 7] *Algorithm 2.1 computes a 2-value-approximate EAT in a dynamic network* $\mathcal{N}$ *with zero transit time in polynomial time complexity.*

3. Approximate EATCF on multi-terminal networks

Pyakurel and Dhamala [13] have solved the earliest arrival transshipment contraflow (EATCF) problem on multi-terminal networks. Their algorithm is based on the MDCF algorithm of [16] for MDCF problem and value-approximate algorithm of [6, 7] for an approximate EAT problem for the arbitrary transit time on each arc of the network. As the solution depends directly upon the size of time expanded network, there exists no polynomial approximation algorithm to solve the 2-value-approximate EATCF problem on multi-terminal networks with arbitrary transit times. Thus, they have claimed that an approximate solution for the EATCF problem can be computed with the modification of their algorithm assuming the transit times zero in each arc of the multi-terminal network $\mathcal{N} = (V, A, b_A, \tau, S, D, \nu(v))$. We extend their claim, in this section, with detail algorithm and proofs.

For the special case of zero transit times, a 2-value-approximate EAT solution is computed in the auxiliary network $\overline{\mathcal{N}} = (V, E, b_E, S, D, \nu(v))$ using the algorithm, Algorithm 3.1. Our algorithm is based on the MDCF algorithm of [16] for the MDCF problem and Algorithm 2.1 of [6, 7] for zero time 2-value approximate EAT problem. In the procedure, the maximum static flow is computed in the auxiliary network $\overline{\mathcal{N}}$ using Algorithm 2.1 and repeated them until a terminal runs out of demand/supply in polynomial time complexity. For more details, we refer to [14].

Algorithm 3.1. *Zero time 2-value-approximate EATCF algorithm*

(1) *Given a network* $\mathcal{N} = (V, A, b_A, \tau, S, D, \nu(v))$.
(2) *Construct the auxiliary network* $\overline{\mathcal{N}} = (V, E, b_E, \tau, S, D, \nu(v))$ *of* $\mathcal{N}$
(3) *Construct the extended network* $\overline{\mathcal{N}}^{\star}$ *of* $\overline{\mathcal{N}}$
(4) *Solve the EAT problem on network* $\overline{\mathcal{N}}^{\star}$ *using Algorithm 2.1 of* [6, 7].
(5) *Arc* $(w, v) \in A$ *is reversed, if and only if the flow along arc* (v, w) *is greater than* $b_A(v, w)$ *or if there is a nonnegative flow along arc* $(v, w) \notin A$.
(6) *Obtain 2-value approximate EATCF solution for the network* $\mathcal{N}$ *with zero transit time.*

To solve the 2-value-approximate EATCF problem with zero transit time with time horizon T, we construct the auxiliary network $\overline{\mathcal{N}}$ first. Then the extended auxiliary network $\overline{\mathcal{N}}^{\star}$ of auxiliary network $\overline{\mathcal{N}}$ is constructed. In Step 4 of Algorithm 3.1, we use Algorithm 2.1 of [6, 7] in extended auxiliary network $\overline{\mathcal{N}}^{\star}$ that contains T copies of each arc and whose supplies/demands are shifted to newly introduced super terminals. This network has a one to one correspondence between an arc copy $e(t)$ and the copy of the arc $(v(t), w(t))$ on time layer t in the time expanded network. In order to prove that the computed flow is a 2-value-approximate EAT on $\overline{\mathcal{N}}^{\star}$, the flow is considered in the residual network of $\overline{\mathcal{N}}^{\star}$ with respect to static flow x_{stat} in which the reverse arcs of the super terminal arcs are deleted. The algorithm performs one MSF calculation per step. The choice of maximal time $a(t)$ guarantees that at least one source or sink runs empty in every iteration and obtains $\nu' = 0$ after δ iterations. Thus a 2-value-approximate EAT on $\overline{\mathcal{N}}^{\star}$ can be computed with at most $\delta \log \nu_{max}$ MSF computations where δ is the number of terminals and $\nu_{max} = \max\{|\nu| \mid \nu \in S \cup D\}$ is the largest supply/demand.

First, we prove that the best possible factor computed by the 2-value approximate EATCF Algorithm 3.1 is 2. For this we give the proof of Lemma 3.2 on the auxiliary network $\overline{\mathcal{N}} = (V, E, b_E, \tau, S, D, \nu(v))$ for the sake of completeness similar to the results in [6, 7] on $\mathcal{N}$.

Lemma 3.2. *Let x_{stat_t} be a maximum flow for time horizon t and let the computed flow by 2-value approximate EATCF algorithm of* [13] *in $\overline{\mathcal{N}}$ be x'_{stat_t}. Then it holds that*

$$val(x_{stat_t}) \leq 2.val(x'_{stat_t}).$$

Proof: First we convert the given network $\mathcal{N}$ into auxiliary network $\overline{\mathcal{N}}$ according to the contraflow configuration. On auxiliary network $\overline{\mathcal{N}}$, we use the 2-value approximate algorithm of [6, 7]. In a step of the algorithm, we compute a difference flow that obtained from subtracting the flow $val(x'_{stat_t})$ from the maximum flow $val(x_{stat_t})$. Let us define the difference flow $x^*_{stat} = (x_{stat_t} - x'_{stat_t})$ that is obtained by sending $(x_{stat_t} - x'_{stat_t})$ on forward arc e, if the value is positive, and sending $-(x_{stat_t} - x'_{stat_t})$ on the backward arc e if the value is negative.

From the difference flow values, we obtain $val(x_{stat_t}) = val(x'_{stat_t}) + val(x^*_{stat})$. The flow x^*_{stat} is valid in $\overline{\mathcal{N}^R_t}$ but not necessarily in $\overline{\mathcal{N}^{R'}_t}$. Let P be any path in the path decomposition of x^*_{stat} which sends an additional unit of flow that is not sent by x'_{stat_t}. As x'_{stat_t} is a maximum flow and the path augmenting algorithm has not found another path, P must be an $s^* - d^*$-path using one of the deleted edges.

However, the total flow value sent through these paths is bounded by the sum of the capacities of the deleted backward edges. This sum is at most $val(x'_{stat_t})$. Thus, we have $val(x^*_{stat}) \leq val(x'_{stat_t})$ and $val(x_{stat_t}) \leq val(x'_{stat_t}) + val(x'_{stat_t})$ and thus, $val(x_{stat_t}) \leq 2.val(x'_{stat_t})$.□

Theorem 3.3. *Algorithm 3.1 computes a 2-value-approximate EATCF solution on $\mathcal{N}$ with zero transit time.*

Proof: Algorithm 3.1 is feasible because of the feasibility of Step 4. Recall that the any approximation solution to an EAT problem with arc reversal on network $\mathcal{N}$ is also a feasible solution to the approximation EAT problem on the auxiliary network $\overline{\mathcal{N}}$. Algorithm 2.1 of [6, 7] and Theorem 2.2 induced a 2-value-approximative EAT solution on $\overline{\mathcal{N}}$. As the amount of flow sent from sources S to sinks D induced from Step 4 is not changed in Step 5, an efficient solution to the 2-value-approximative EATCF problem on $\mathcal{N}$ is obtained. □

Corollary 3.4. *Algorithm 3.1 computes a 2-value-approximate EATCF solution with zero transit time in polynomial time complexity.*

Proof: As a MSF solution is computed in each time period in Step 4 of Algorithm 3.1, the complexity is dominated by the complexity of MSF computation. It simply concludes that the complexity of the algorithm is bounded by polynomial time. □

4. Conclusions

We solved the approximate earliest arrival transshipment contraflow problem on multi-terminal networks in polynomial time complexity that had been claimed in [13]. Although the problem was solved in the same complexity as without contraflow, the flow value computed by contraflow model increases significantly. From the analytical point of view, it has been realized that flow value may be doubled for given time horizon. Moreover, the time needed to transship given amount of flow value will be at most half with contraflow configuration.

To the best of our knowledge, the problem we solved is for the first time on complex contraflow networks using discrete time setting. However, it is still unsolved problem whether the earliest arrival transshipment contraflow problem on multi-terminal networks is polynomially solvable.

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