IMPROVED UPPER BOUND TO PRODUCT RATE VARIATION PROBLEM

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Abstract: The sequencing problem which minimizes the deviation between the actual (integral) and the ideal (rational) cumulative production of a variety of models of a common base product is called the product rate variation problem. If the objective is to minimize the maximum deviation, the problem is bottleneck product rate variation problem and the problem with the objective of minimizing all the deviations is the total product rate variation problem.

The problem has been widely studied with several pseudo-polynomial time exact algorithms and heuristics. The lower bound of a feasible solution to the problem has been investigated to be tight. However, the upper bound of a feasible solution had been established in the literature which could further be improved. In this paper, we propose the improved upper bound for BPRVP and TPRVP.

Key Words: Bound, Product rate variation problem, Non-linear integer programming problem

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1. Introduction

The product rate variation problem has been developed when the manufacturing systems switched from mass production to mass customization in order to address customer's demand of a variety of products of a common base product at reasonable prices. The problem addresses sequencing of products in the assembly line, a part of a commonly known Toyota production system [14, 15] in order to eliminate inventories and shortages. The challenge of this just-in-time approach is to prevent shortages while keeping the inventory at a minimum level. Empirical observations have confirmed the positive effects of this approach of mixed-model just-in-time (MMJIT) systems in manufacturing companies [7] and also see [13]. The sequence planning is an important issue in mixed-model assembly lines and is closely linked to the complexity of the system [19]. A group decision making approach has been recently proposed in manufacturing systems, [18].

There exist two variants of the sequencing problem of MMJIT systems. One the bottleneck product rate variation problem (BPRVP) with the objective of minimizing the maximum deviation and the other the total product rate variation problem (TPRVP) with the objective of minimizing all the deviations between the actual and the ideal cumulative

productions [13]. Both variants of the problem have strong mathematical base with wide real-world applications, see [3], a survey and therein.

The BPRVP is reduced into a perfect matching problem for a feasible solution and a bisection search algorithm is used for its optimal solution. The solution procedure has been improved with an improved upper bound and the necessary and sufficient condition for the existence of a feasible solution [2]. The PRVP with the objective function other than the absolute deviation has been solved in [4] and [8] by investigating the upper bounds corresponding to the objective functions. The problem with different objective functions yields solution in the same complexity $O(DlogD)$, where D stands for the total demand of all the models, of the problem with an absolute deviation objective [4, 8, 16]. An improved upper bound to the BPRVP has been investigated in [10]. Moreover, the problem with the asymmetric objective function that means different penalty for the inventory and for the shortage has been established [9]. The penalty for tardiness is considered to be higher than the penalty for earliness with the assumption that frequent and extensive shortages are likely to threaten the trust of the customers in the ability of a company to deliver.

The TPRVP is also well studied with exact solution method which is firstly reduced into an assignment problem and is solved in pseudo-polynomial time and also with some heuristics [1, 4, 5, 6, 12, 13, 17]. The complexity of the exact solution method is D^3, where D is total demand of all copies. The lower and the upper bounds of the TPRVP have also been established, [11]. However, the upper bound can further be improved.

The contribution proposed in this paper can be applied in developing production sequences in the assembly lines in which a variety of models of a common base product is produced. The sequencing approach of production with the implementation of this investigation is exact and based on some assumptions.

Let us consider total demand D of $n, n \geq 2$, different models with d_i copies of model $i, i = 1, 2, \ldots, n$ of a common base product. We assume equal processing time of each model $i, i = 1, 2, \ldots, n$, in total D equal time units which is called time horizon. A copy of a model is produced in a time unit k means that the copy of the model is produced during the time period from $k - 1$ to $k, k = 1, 2, \ldots, D$. Consider $r_i = \frac{d_i}{D}$ to be the demand rate and x_{ik} and $kr_i, i = 1, 2, \ldots, n; k = 1, 2, \ldots, D$, to be the actual and the ideal cumulative productions, respectively, of model i produced during the time units 1 through k. Let us assume equal cost for both inventory $x_{ik} - kr_i > 0$, and shortage $kr_i - x_{ik} > 0$. A non linear integer programming formulation for the PRVP,[13, 16] is given below.

Minimize

$$F_m = \max_{ik} |x_{ik} - kr_i|^m \qquad \text{for BPRVP}$$

Minimize

$$\widetilde{F}_m = \sum_{k=1}^{D} \sum_{i=1}^{n} |x_{ik} - kr_i|^m \qquad \text{for TPRVP}$$

subject to

$$\sum_{i=1}^{n} x_{ik} = k, \quad k = 1, 2, \ldots, D$$

$$x_{i(k-1)} \leq x_{ik}, \quad i = 1, 2, \ldots, n; \quad k = 2, 3, \ldots, D$$

$$x_{iD} = d_i, x_{i0} = 0, \quad i = 1, 2, \ldots, n$$

$$x_{ik} \geq 0, \text{integer}, \quad i = 1, 2, \ldots, n; \quad k = 1, 2, \ldots, D .$$

In this paper, we propose the improved upper bounds to the BPRVP and the TPRVP both. The remainder of the paper is structured as follows. In Section 2, we discuss the level curves, lower bound and improved upper bounds for both BPRVP and TPRVP as the main contribution of the paper. The last section concludes the paper.

2. Bounds

Level curve

The actual cumulative production $x_{ik}, i = 1, 2, \ldots, n; k = 1, 2, \ldots, D$, a sequence-dependent integer, has nD values with $x_{ik} \in \{j | j = 0, 1, 2, \ldots, d_i; i = 1, 2, \ldots, n\}$. However, the ideal cumulative production $kr_i, i = 1, 2, \ldots, n; k = 1, 2, \ldots, D$, a sequence-independent rational, is such that $kr_i \in \{\frac{d_i}{D}, \frac{2d_i}{D}, \ldots, d_i\}, i = 1, \ldots, n$. Let $j, j = 1, \ldots, d_i$, be the number of copies of a model $i, i = 1, 2, \ldots, n$, and (i, j) be the jth copy of model $i, i = 1, 2, \ldots, n$. There exist at most $n + D$ different values of x_{ik}. Hence, one can replace x_{ik} by j with $j = 0, 1, \ldots, d_i; i = 1, 2, \ldots, n$, in the level curve (see Fig. 1) of the objective value of the function. The level curve for copy (i, j) of the objective function of the PRVP is defined as

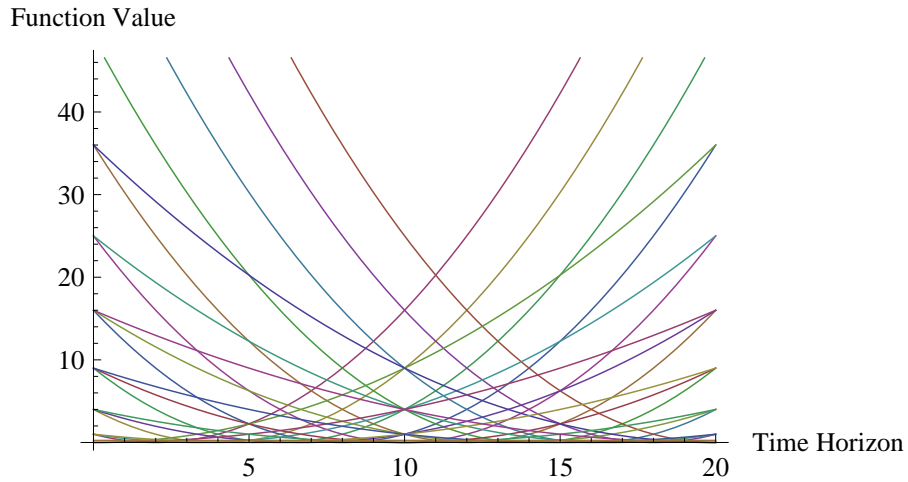$$f_{ij}^m = |j - kr_i|^m , i = 1, 2, \ldots, n; j = 0, 1, \ldots, d_i .$$

FIGURE 1. Level curves f_{ij}^2 for the instance $(d_1 = 4, d_2 = 6, d_3 = 10)$

The horizontal line with the value $B > 0$ intersecting the level curve for each copy $(i, j), i = 1, 2, \ldots, n; j = 1, 2, \ldots, d_i,$ is called a bound for the problem.

Lower bound

The importance of a lower bound LB_m on the optimal value of function F_m for the problem results from the fact that a feasible sequence s of an instance $(d_1, d_2, \ldots, d_n)$ obtained with an objective function value equal to the lower bound is optimal. However, not all instances are even feasible at this value. The tight lower bound for BPRVP has been investigated to be $LB_m = (1 - r_{max})^m$, see [4, 8]. It is note-worthy that a lower bound on the absolute deviation objective function for the BPRVP has been established by Steiner and Yeomans [16]. Likewise the lower bound for TPRVP has been investigated to be $LB_m = nD(1 - r_{max})^m$, see [11].

Upper bound

An upper bound UB_m, m being a positive integer, must satisfy the following two inequalities which are the necessary and sufficient condition for the existence of a feasible sequence for the BPRVP with the objective function F_m ,

$$(2.1) \qquad \sum_{i=1}^{n}(\left\lfloor k_2 r_i + \sqrt[m]{UB_m} \right\rfloor - \left\lceil (k_1 - 1)r_i - \sqrt[m]{UB_m} \right\rceil) \geq k_2 - k_1 + 1$$

and

$$(2.2) \qquad \sum_{i=1}^{n}(\left\lceil k_2 r_i - \sqrt[m]{UB_m} \right\rceil - \left\lfloor (k_1 - 1)r_i + \sqrt[m]{UB_m} \right\rfloor) \leq k_2 - k_1 + 1$$

with $k_1, k_2 \in \{1, \ldots, D\}, k_1 \leq k_2$, see Khadka [8]. If $m = 1$, the two inequalities become the necessary and sufficient condition for the problem with the objective function F_1, see Brauner and Crama [2].

Initially the upper bound UB_1 to the problem BPRVP has been investigated to be $UB_1 = 1$, [16] and later it has been improved to be $UB_1 = 1 - \frac{1}{D}$ for the problem with the objective function F_1, [2] and to be $UB_m = (1 - \frac{1}{D})^m$ for the problem with the objective function F_m, [8]. The upper bound $UB_m = (1 - \frac{1}{D})^m$ can be improved to be $UB_m = (1 - r_{min})^m$, where $r_{min} = \frac{d_{min}}{D}$, in the case of an instance $(d_1, d_2, \ldots, d_n)$ with $d_{min} \geq 1$, $\min_i(d_i) = d_{min}, i = 1, 2, \ldots, n$.

Theorem 2.1. *Let*

$$UB_m = (1 - r_{min})^m, \text{ where } r_{min} = \frac{d_{min}}{D}, \min_i(d_i) = d_{min}, i = 1, 2, \ldots, n .$$

Then UB_m is an upper bound on the largest value of the objective function F_m of a feasible solution for BPRVP.

Proof. Let us consider a general instance $(d_1, d_2, \ldots, d_n)$ with $d_{min} \geq 1$, where $\min_i(d_i) = d_{min}, i = 1, 2, \ldots, n$. If UB_m is an upper bound on the largest value of the objective function F_m of a feasible solution for the BPRVP, then this bound UB_m satisfies the inequality

$$(2.3) \qquad\qquad\qquad |x_{ik} - kr_i|^m \leq UB_m,$$

for all $i = 1, 2, \ldots, n; k = 1, 2, \ldots, D$, for any feasible solution $s \in \mathcal{X}$.

We claim that the improved upper bound on the largest value of the objective function F_m is

$$(2.4) \qquad\qquad UB_m = \left(1 - r_{min}\right)^m,$$

where

$$\min_i(r_i) = r_{min} = \frac{d_{min}}{D}, i = 1, 2, \ldots, n.$$

We need to show the proposed upper bound (2.4) satisfying the two inequalities (2.1) and (2.2).

We can write

$$\left\lfloor k_j r_i + \sqrt[m]{UB_m} \right\rfloor = \lfloor k_j r_i + 1 - r_{min} \rfloor.$$

$$= \left\lfloor k_j r_i + 1 - \frac{d_{min}}{D} \right\rfloor, i = 1, 2, \ldots, n; \ j = 1, 2.$$

If $k_j r_i$ is an integer,

$$\left\lfloor k_j r_i + 1 - \frac{d_{min}}{D} \right\rfloor = k_j r_i, \ i = 1, 2, \ldots, n; j = 1, 2,$$

and if $k_j r_i$ is not an integer,

$$k_j r_i = \lfloor k_j r_i \rfloor + (k_j r_i)_{frac}, \ i = 1, 2, \ldots, n; j = 1, 2,$$

where $(k_j r_i)_{frac}$ is the fractional part of $k_j r_i$.
Since $\frac{d_{min}}{D} \le (k_j r_i)_{frac} \le 1 - \frac{d_{min}}{D}$,

$$\lfloor k_j r_i + 1 - r_{min} \rfloor = \left\lfloor \lfloor k_j r_i \rfloor + (k_j r_i)_{frac} + 1 - \frac{d_{min}}{D} \right\rfloor$$

$$\ge \lfloor k_j r_i \rfloor + 1$$

$$> k_j r_i, \ i = 1, 2, \ldots, n; j = 1, 2.$$

Therefore,

$$(2.5) \qquad\qquad \left\lfloor k_j r_i + \sqrt[m]{UB_m} \right\rfloor \ge k_j r_i, \ i = 1, 2, \ldots, n; j = 1, 2.$$

Likewise,

$$\left\lceil k_j r_i - \sqrt[m]{UB_m} \right\rceil = \lceil k_j r_i - 1 + r_{min} \rceil.$$

$$= \left\lceil k_j r_i - 1 + \frac{d_{min}}{D} \right\rceil, i = 1, 2, \ldots, n; j = 1, 2.$$

If $k_j r_i$ is an integer,

$$\left\lceil k_j r_i - 1 + \frac{d_{min}}{D} \right\rceil = k_j r_i, \ i = 1, 2, \ldots, n; j = 1, 2,$$

and if $k_j r_i$ is not an integer,

$$\left\lceil k_j r_i - 1 + \frac{d_{min}}{D} \right\rceil = \left\lceil \lfloor k_j r_i \rfloor + (k_j r_i)_{frac} - 1 + \frac{d_{min}}{D} \right\rceil$$

$$\le \lceil \lfloor k_j r_i \rfloor \rceil$$

$$< k_j r_i, \ i = 1, 2, \ldots, n; j = 1, 2.$$

Therefore,

$$(2.6) \qquad \left\lceil k_j r_i - \sqrt[m]{UB_m} \right\rceil \leq k_j r_i, \ i = 1, 2, \ldots, n; j = 1, 2 \ .$$

Hence, using the inequalities (2.5) and (2.6),

$$\sum_{i=1}^{n} (\left\lfloor k_2 r_i + \sqrt[m]{UB_m} \right\rfloor - \left\lceil (k_1 - 1) r_i - \sqrt[m]{UB_m} \right\rceil) \ \geq \ \sum_{i=1}^{n} k_2 r_i - \sum_{i=1}^{n} (k_1 - 1) r_i$$
$$\geq \ k_2 - k_1 + 1 \ .$$

And,

$$\sum_{i=1}^{n} (\left\lceil k_2 r_i - \sqrt[m]{UB_m} \right\rceil - \left\lfloor (k_1 - 1) r_i + \sqrt[m]{UB_m} \right\rfloor) \ \leq \ \sum_{i=1}^{n} k_2 r_i - \sum_{i=1}^{n} (k_1 - 1) r_i$$
$$\leq \ k_2 - k_1 + 1 \ .$$

This shows that

$$UB_m = (1 - r_{min})^m$$

satisfies the two inequalities (2.1) and (2.2) proving to be the upper bound of a feasible solution for the BPRVP with the objective function

$$\max_{i,k} \ |x_{ik} - k r_i|^m, i = 1, 2, \ldots, n; \ k = 1, 2, \ldots, D \ .$$

$\square$

In particular, for any instance $(d_1, d_2, \ldots, d_n)$ with $d_{min} = 1$, the upper bound,

$$UB_m = (1 - r_{min})^m \ ,$$

is exactly the same as the upper bound established in [2] for the problem with the objective function F_1 and, in [8] for the objective function F_m, m being a positive integer.

The improved upper bound for TPRVP can be established to be

$$UB_m = nD(1 - r_{min})^m$$

, which has been proven below.

Theorem 2.2. *Let*

$$\widetilde{UB}_m = nD(1 - r_{min})^m \ , r_{min} = \frac{d_{min}}{D}, min(d_i) = d_{min}, i = 1, 2, \ldots, n.$$

Then $\widetilde{UB}_m$ is an upper bound on the largest value of the objective function $\widetilde{F}_m$ of a feasible solution for TPRVP.

Proof. If $\widetilde{UB}_m$ is an upper bound on the largest value of the objective function $\widetilde{F}_m$ of a feasible solution for TPRVP, then this bound $\widetilde{UB}_m$ satisfies the inequality

$$(2.7) \qquad \sum_{k=1}^{D} \sum_{i=1}^{n} |x_{ik} - k r_i|^m \leq \widetilde{UB}_m$$

for any feasible solution $\mathbf{x} \in \mathcal{X}$.

It has been established that the improved upper bound is $(1 - r_{min})^m$ for BPRVP with the objective function F_m.

Now, we obtain

$$
\begin{aligned}
\sum_{k=1}^{D}\sum_{i=1}^{n}|x_{ik}-kr_i|^m &= \sum_{i=1}^{n}|x_{i1}-1r_i|^m+\cdots+\sum_{i=1}^{n}|x_{iD}-Dr_i|^m \\
&= |x_{11}-1r_1|^m+\cdots+|x_{n1}-1r_n|^m+\ldots \\
&+ |x_{1D}-Dr_1|^m+\cdots+|x_{nD}-Dr_n|^m \\
&\leq nD\cdot\max_{i,k}|x_{ik}-kr_i|^m \\
&\leq nD\left(1-r_{min}\right)^m .
\end{aligned}
$$

Hence, an upper bound $\widetilde{UB}_m$ on the largest value of the objective function $\widetilde{F}_m$ of a feasible solution for TPRVP is given by

$$
\widetilde{UB}_m = nD\left(1-r_{min}\right)^m .
$$

$\square$

3. Concluding Remarks

We considered the product rate variation problem with the unimodular symmetric objective function having zero at zero. It has two variants one the problem which is called the bottleneck product rate variation problem with the objective F_m, m being a positive integer, of minimizing its maximum deviation between the actual and the ideal cumulative production of a variety of models of a common base product which has been solved with an exact solution having its complexity $DlogD$, D being the total demand. The other variant is known as total product rate variation problem with the objective function $\widetilde{F}_m$ of minimizing all the deviations instead of its maximum deviation. This problem has been solved with many heuristics and an exact solution with complexity D^3, [12].

A lower bound $1-r_{max}$ of a feasible solution of BPRVP with the objective function F_1, [16], and $(1-r_{max})^m$ for the objective function F_m, m being a positive integer, [8], and an upper bound $1-\frac{1}{D}$ for the objective function F_1, [2], and $(1-\frac{1}{D})^m$, [8], for the objective function F_m of a feasible solution for the BPRVP have been established.

Likewise, a lower bound and an upper bound of a feasible solution of TPRVP $nD(1-r_{max})^m$ and $nD\left(1-\frac{1}{D}\right)^m$, m being a positive integer investigated in [11].

In this paper, we have proposed improved upper bounds to be $(1-r_{min})^m$ and $nD(1-r_{min})^m$ for BPRVP and TPRVP, respectively. Implementation of the improved upper bounds while applying the exisitng algorithms to find optimal solution would be interesting in the future work.

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