The Nepali Mathematical Sciences Report, Vol. 37, No. 1 & 2, 2020:63-70
DOI: https://doi.org/10.3126/nmsr.v37i1-2.34092

APPROXIMATING THE SUM OF INFINITE SERIES OF NON NEGATIVE TERMS WITH REFERENCE TO INTEGRAL TEST

DAYA RAM PAUDYAL

Birendra Multiple Campus, Tribhuvan University, Kathmandu, Nepal

Correspondence to: Daya Ram Paudyal,
Email: dayaram_paudyal@yahoo.com

Abstract: This paper describes a method of obtaining approximate sum of infinite series of positive terms by using integrals under its historical background. It has shown the application of improper integrals to determine whether the given infinite series is convergent or divergent. Here, the limits of the integrals and the series usually extend to infinity though they may be slowly convergent. We have also established a relation to approximate the sum of infinite series of positive terms with a suitable example.

Keywords: Convergence, Decreasing, Infinite series, Partial Sum.
AMS (MOS) Subject Classification. MSC: 40C10

1. Introduction

Infinite series have great role in modern arena of mathematics. Out of infinite series existing in different branches of mathematics and physics, some are convergent and some are divergent. Even when convergent, a series may converge too slowly to have the fixed sum. For the determination of whether the given series is convergent or divergent there are various methods. But it is little be difficult to get the actual sum. Here we would like to connect the concept of integrals to determine whether the given infinite series of positive terms is convergent or not with their cor-responding sum. If the series is asymptotic, the magnitude of the smallest term seems to set a lower bound on the accuracy of evaluation. Madhava (c.1344-c.1425), an Indian mathematician, studied the mathematical series and obtained the expansion of sinx, cosx, tanx and Arctanx as the infinite series for the first time in the world [1]. Pietro Mengoli studied the different infinite series and established the sum of some infinite series in finite value. Bonaventura cacalieni (Ital-ian), Wallis (English), James Gregory (Scottish) and others also studied about the different types of infinite series. Jacob Bernoulli, Daniel Bernoulli, Johan Bernoulli and Leonhard Euler did much work in infinite series and their sums in 18th century [2]. In the same century Leibnitz and Newton gave a very strong impulse in the theory of infinite series. Consequently they established the foundation of calculus. Rigorous foundation of theory of infinite series was laid in the first half of the 19th century Ernesto Cesaro (Italian), Emile Borel (French), Leopold Fejer (Hungarian) and other established some theories to find the sum of infinite series. In the 20th century Godfrey Harold Hardey and John Edensor Littlewood including the Indian mathematician Ramanujan brought different ideas in summing infinite series whether they are convergent or divergent. Especially, they gave different ideas of summability methods of divergent series. The concept of infinite sum of positive terms is mysterious and very interesting. Pietro Mengoli (1626-1686) used infinite series to good effect in Novae quadraturae arithmeticae, sue de additione fractionum published in Bologna in 1650, developing ideas which had first been investigated by Cataldi (1548-1626) [3]. He begins with the summation

Received: May 9, 2020, Accepted/Published Online: January, 2021 .

of geometric series, and then shows that the harmonic series does not converge. In doing so he became the first person to prove that it was possible for a series whose terms tend to zero to be made larger than any given number. He also investigated the harmonic series with alternating signs converges to log (2). In the series of studying the sum of infinite series he became unable to find the sum of the infinite series

$$\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+... \tag{1.1}$$

This series 1.1 is popularly known as Basel problem in the memory of Bernoulli brothers and Leonhard Euler who were born in Basel. According to history of mathematics the first world class mathematicians to work on the Basel Problem were the Swiss brothers Jacob Bernoulli (1654-1705) and Johann Bernoulli (1667-1748). The Bernoulli brothers had been among the first to understand and apply the newly invented calculus, which they learned from calculus's co-inventor, Gottfried Leibniz (1646-1716). By the 1690s, they were considered to be the leading mathematicians in Europe. Over a period of many years both Bernoulli brothers tried to solve the Basel problem, probably motivated in part by a desire to best his rival, but they had no success. Leibniz also worked on the problem for years and got no solution [3]. The problem had first been posed in 1644 by Pietro Mengoli (1626-1686). The problem was popularized by the celebrated mathematician Jacob Bernoulli in 1689. John Wallis calculated its value of its first three decimals (1.645) [4]. Then, in 1735, when he was 28, Euler announced that he had found a solution. He reached his insightful conclusion that

$$\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+...=\frac{\pi^2}{6} \tag{1.2}$$

Euler did many works in the theory of infinite series. Euler must be regarded as the first master of the theory of infinite series [4].

2. Preliminaries

2.1. **Definition.** Let $\sum_{n=0}^{\infty} a_n$ be an infinite series having nth partial sum $S_n = \sum_{k=1}^{n} a_k$. Such infinite series is said to be convergent to the sum S if $\lim\limits_{n\to\infty} s_n$ exists and equal to S where S_n is the sequence of partial sums. If the sequence of partial sums has no limit then the series is said to be divergent [5]. For example, consider a series like

$$\sum_{k=1}^{\infty}\frac{1}{2^{k-1}}=1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+... \tag{2.1}$$

If we calculate the partial sums of it we get the scenario as Table 1. From this Table 1 we obtained that there exists a sequence of partial sums whose nth term is given by $S_n = 2 - \frac{1}{2^{n-1}}$. Then the sum of the given series is given by

$$\begin{aligned} S &= \sum_{k=1}^{\infty}\frac{1}{2^{k-1}} \\ &= \lim_{n\to\infty}(2-\frac{1}{2^{n-1}}) = 2 \end{aligned}$$

Table 1. Algorithmic approach for partial sum

No of terms	Partial sum	Sum(value)	Trend of partial sums
1	$S_1 = 1$	1	2-1
2	$S_2 = 1 + \frac{1}{2}$	$\frac{3}{2}$	2- $\frac{1}{2}$
3	$S_3 = 1 + \frac{1}{2} + \frac{1}{4}$	$\frac{7}{4}$	2-$\frac{1}{4}$
4	$S_4 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$	$\frac{15}{8}$	2-$\frac{1}{8}$
.	.		
.	.		
.	.		
n	$S_n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + ... + \frac{1}{n}$	$\frac{2^n-1}{2^{n-1}}$	2-$\frac{1}{2^{n-1}}$

It means the limit exists and hence this infinite series of positive terms is convergent and converges to 2. Furthermore, we can write

$$\begin{aligned} 2=2 \implies 2=1+1 \implies 2=1+\frac{1}{2}+\frac{1}{2} \implies 2=1+\frac{1}{2}+\frac{1}{4}+\frac{1}{4} \\ \implies 2=1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{8} \\ \implies 2=1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{16} \\ \implies 2=1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{32} \\ \implies 2=1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\frac{1}{64} \\ \implies 2=\frac{1}{2^0}+\frac{1}{2^1}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\frac{1}{2^5}+\frac{1}{2^6}+....+\frac{1}{2^{n-1}}+\frac{1}{2^{n-1}} \end{aligned}$$

If we extend in this way we get an infinite series and it also suggests us that the sum of series 2.1 is 2 [6].

2.2. **Some Related Theorems.**

2.2.1. *Theorem.* A series $\sum_{n=1}^{\infty} a_n$ of non-negative terms converges if and only if its partial sums are bounded from above.

Proof:

Let $\{S_n\}$ be the sequence of the partial sums of the terms of the given infinite series of non negative terms. Suppose it is bounded above by some constant number M. Our task is to prove the series $\sum_{n=1}^{\infty} a_n$ is convergent. For this let N be the least upper bound of the sequence which is bounded above by M. Now plot the points $(1, S_1), (2, S_2), (3, S_3), (4, S_4), ..., (n, S_n), (n+1, S_{n+1}), ...$ in XY plane as shown in the fig. 2. We see all the points are below the line y=L. For any given $\varepsilon > 0$ there exists some positive integer K such that some of points (n, S_n) lie above the line $y = L - \varepsilon$ for $n > K$. Thus there are infinitely many points of (S_n) that lie in the neighbourhood of L and hence its limit exists and it is equal to L. It means the limit of the sequence$\{S_n\}$ of partial sums exists.

So, by definition, the series $\sum_{n=1}^{\infty} a_n$ is convergent. Conversely let $\sum_{n=1}^{\infty} a_n$ is a convergent series then we have to show that the sequence (S_n) of its partial sums is bounded above. By definition of convergent series,

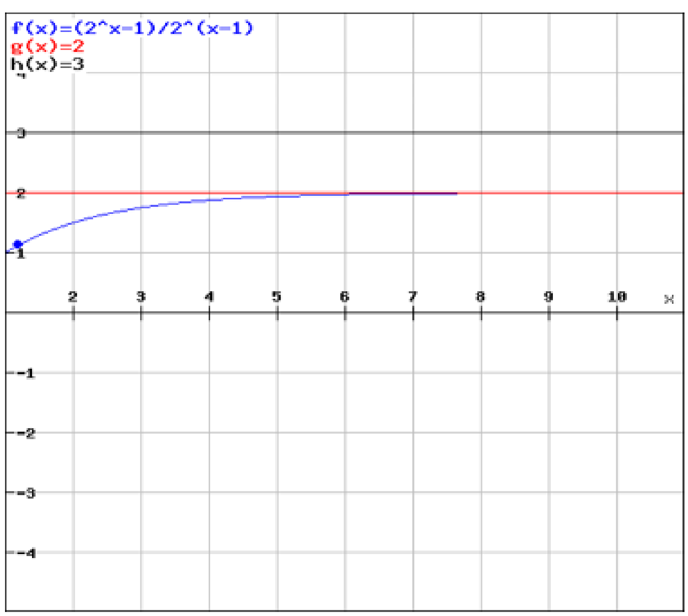

FIGURE 1. Representation of partial sums

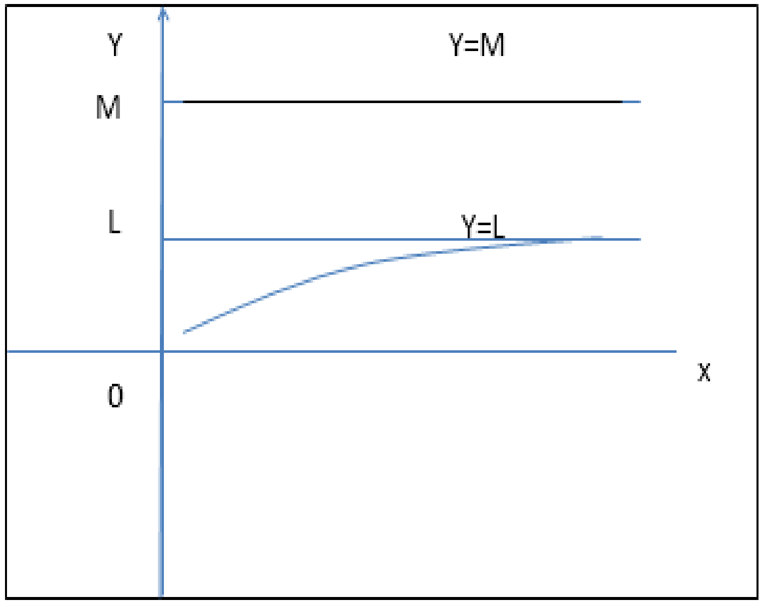

FIGURE 2. Trend of convergence sequence.

we must have the limit of sequence of partial sums. So let L be its limit. Therefore, $\lim_{n\to\infty}\{S_n\} = L$.It means for given $\varepsilon > 0$, there exists a positive number K_1 such that $n \geq K_1$ implies that $|S_n\text{-L}| < \varepsilon \implies$ L-ε $< S_n < L + \varepsilon$. As the sequence $\{S_n\}$ is non-decreasing, there is no points of $\{S_n\}$ which lies between L and L+ε and hence L is the least upper bound of $\{S_n\}$. So it must have an upper bound. Thus $\{S_n\}$ is bounded above. This completes the proof.

2.2.2. *Theorem(Integral test).* Let $\{a_n\}$ be a sequence of positive terms. Suppose a_n=f(n), where f is continuous, positive,decreasing function of x for all $x \geq N$,a positive integer.Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)dx$ both converge or both diverge[7].
Proof:
Let $\{a_n\}$=$\{$f(n)$\}$ be a sequence of monotonically decreasing positive terms. Then the sum of all its terms is defined by the infinite series $\sum_{n=1}^{\infty} a_n$. Let f(n)=a_n for all n and construct a figures as shown in fig. 3 and fig. 4. From fig.3, it is seen that the total area covered by the rectangles is more than the area under the curve $y = f(x)$ from $x = 1$ to $x = n + 1$. So, we have

$$\int_1^{n+1} f(x)dx \leq a_1 + a_2 + a_3 + ... + a_n \tag{2.2}$$

Similarly from fig. 4, we get

$$a_1 + a_2 + a_3 + ... + a_n \leq \int_1^{n} f(x)dx \tag{2.3}$$

From relation 2.2 and 2.3 we obtained

$$\int_1^{n+1} f(x)dx \leq a_1 + a_2 + a_3 + ... + a_n \leq \int_1^{n} f(x)dx \tag{2.4}$$

Thus this inequality 2.4 suggests that if$\int_1^{n} f(x)dx$ is finite then the series $\sum a_n$ is finite and if $\int_1^{n+1} f(x)dx$ is infinite then $\sum a_n$ is also infinite. This concept can be extended for $n = N$ to $n = \infty$ to conform that $\sum_{n=N}^{\infty} a_n$ converges while $\int_{n=N}^{\infty} f(x)dx$ converges [8]. This completes the proof.

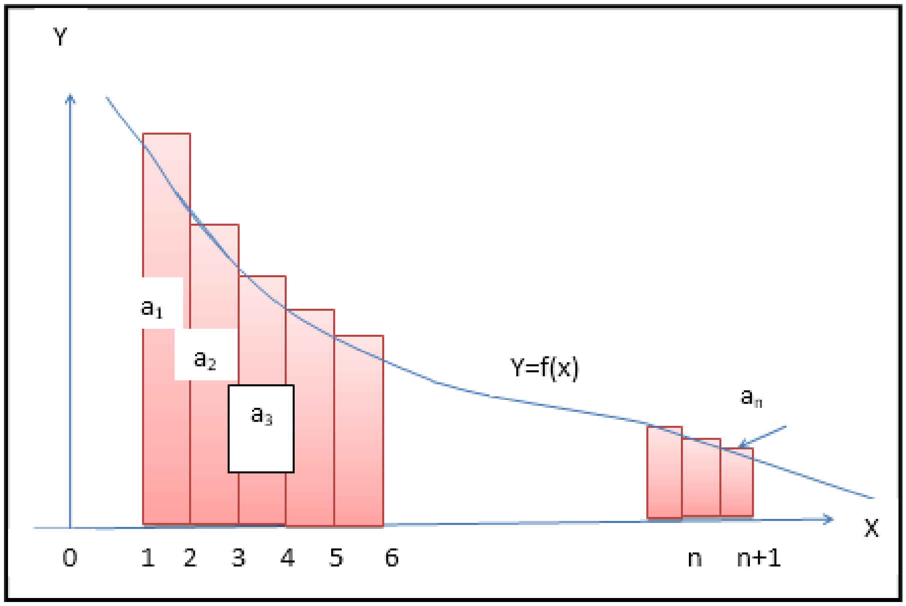

FIGURE 3. Area: More than the area under the curve (Upper Riemann Sum).

3. APPLICATION OF INTEGRAL TEST

Consider the series discussed in 2.1 given by

$$\sum_{k=1}^{\infty} \frac{1}{2^{k-1}} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ...$$

now we would like to apply the integral test to find the sum of 2.1 which is a geometric series and its sum is given by $\frac{1}{1-\frac{1}{2}} = 2$. Our concern is weather this sum is valid or not by using improper integral as disused below. Let $n = N$, and f(x)=$\frac{1}{2^{x-1}}$ then

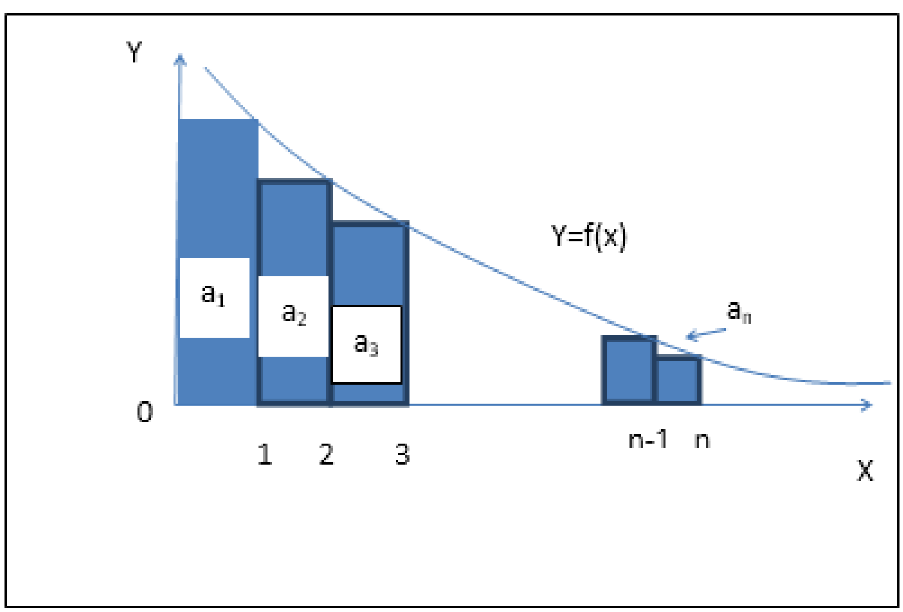

FIGURE 4. Area: Less than the area under the curve (Lower Riemann Sum).

$$\int_{n=N}^{\infty} f(x)dx = \int_{N}^{\infty} \frac{1}{2^{x-1}}dx$$
$$= 2\int_{N}^{\infty} 2^{-x}dx$$
$$= 2\lim_{t\to\infty}\int_{N}^{t} 2^{-x}dx$$
$$= 2\lim_{t\to\infty}\left[\frac{2^{-x}}{-ln2}\right]_{N}^{t}$$
$$= 2\lim_{t\to\infty}\left[\frac{2^{-t}}{-ln2} + \frac{2^{-N}}{ln2}\right]_{N}^{t}$$
$$= \frac{1}{2^{N-1}ln2}$$

which is finite value for every $N = 1, 2, 3,...$
It means the sum of infinite series $\sum_{k=1}^{\infty} \frac{1}{2^{k-1}}$ exists and is equal to some finite value[9]. Now we relate the sum with this integral as shown in the following Table2.

The sum of this series 2.1 is 2 and by analyzing this Table 2, we concluded that the actual sum always lies

TABLE 2. Algorithmic approach for proving the theorem.

N	a_1+a_2+a_3+...+a_n +$\int_n^\infty f(x)dx$	a_1+a_2+a_3+...+a_n +$\int_{n+1}^\infty f(x)dx$	Error
1	$1+\int_1^\infty \frac{1}{2^{x-1}}dx = 2.442695041$	$1+\int_2^\infty \frac{1}{2^{x-1}}dx = 1.72134752$	0.0820212805
2	$1+\frac{1}{2}+\int_2^\infty \frac{1}{2^{x-1}}dx = 2.22134752$	$1+\frac{1}{2}+\int_3^\infty \frac{1}{2^{x-1}}dx = 1.86067376$	0.04101064
3	$\sum_{k=1}^{3} \frac{1}{2^{x-1}}+\int_3^\infty \frac{1}{2^{x-1}}dx$=2.11067376	$\sum_{k=1}^{3} \frac{1}{2^{x-1}}+\int_4^\infty \frac{1}{2^{x-1}}dx$=1.93033688	0.02050532
4	$\sum_{k=1}^{4} \frac{1}{2^{x-1}}+\int_4^\infty \frac{1}{2^{x-1}}dx$=2.05533688	$\sum_{k=1}^{4} \frac{1}{2^{x-1}}+\int_5^\infty \frac{1}{2^{x-1}}dx$=1.96516844	0.01042686
5	$\sum_{k=1}^{5} \frac{1}{2^{x-1}}+\int_5^\infty \frac{1}{2^{x-1}}dx$=2.02766844	$\sum_{k=1}^{5} \frac{1}{2^{x-1}}+\int_6^\infty \frac{1}{2^{x-1}}dx$=1.9825842	0.00512632
6	$\sum_{k=1}^{6} \frac{1}{2^{x-1}}+\int_6^\infty \frac{1}{2^{x-1}}dx$=2.01383422	$\sum_{k=1}^{6} \frac{1}{2^{x-1}}+\int_7^\infty \frac{1}{2^{x-1}}dx$=1.99129211	0.002563165
7	$\sum_{k=1}^{7} \frac{1}{2^{x-1}}+\int_7^\infty \frac{1}{2^{x-1}}dx$=2.00691711	$\sum_{k=1}^{7} \frac{1}{2^{x-1}}+\int_8^\infty \frac{1}{2^{x-1}}dx$=1.995646055	0.001281555
8	$\sum_{k=1}^{8} \frac{1}{2^{x-1}}+\int_8^\infty \frac{1}{2^{x-1}}dx$=2.003458555	$\sum_{k=1}^{8} \frac{1}{2^{x-1}}+\int_9^\infty \frac{1}{2^{x-1}}dx$=1.997823028	0.0006407915
9	$\sum_{k=1}^{9} \frac{1}{2^{x-1}}+\int_9^\infty \frac{1}{2^{x-1}}dx$=2.001729278	$\sum_{k=1}^{9} \frac{1}{2^{x-1}}+\int_{10}^\infty \frac{1}{2^{x-1}}dx$=1.998911514	0.000345396
10	$\sum_{k=1}^{10} \frac{1}{2^{x-1}}+\int_{10}^\infty \frac{1}{2^{x-1}}dx$=2.000864639	$\sum_{k=1}^{10} \frac{1}{2^{x-1}}+\int_{11}^\infty \frac{1}{2^{x-1}}dx$=1.999455757	0.000160197

between lower sum and upper sum. If the sum is assumed as S it is observe that

$$a_1 + a_2 + a_3 + ... + a_n + \int_{n+1}^{\infty} f(x)dx \le S \le a_1 + a_2 + a_3 + ... + a_n + \int_{n}^{\infty} f(x)dx$$

, i.e.,

$$\sum_{k=1}^{n} a_k + \int_{n+1}^{\infty} f(x)dx \le S \le \sum_{k=1}^{n} a_k + \int_{n}^{\infty} f(x)dx \tag{3.1}$$

Even then if we take n small then there exists more error in comparison to the large value of n. So in particular case we have developed the error given by

$$E_n = \frac{3 - ln16}{2^n ln4} \tag{3.2}$$

$$\lim_{n\to\infty} E_n = \lim_{n\to\infty} \frac{3 - ln16}{2^n ln4} \tag{3.3}$$

The limit obviously becomes zero. It means if we increase the value of n then the error decreases rapidly and if it goes to sufficiently large the error vanishes. In such cases the sum is equal to $S = 2$. From this analysis we also declare that $\sum_{n=1}^{\infty} a_n$ is not equal to $\int_1^\infty f(x)dx$ but this improper integral is very useful to estimate the value of sum of this type of infinite series [10]. In this direction many theorems have been established by many mathematicians [11, 12, 13, 14, 15].

4. Main Theorem

Theorem: Let $\sum_{n=1}^{\infty} a^{1-x}, a > 1$ be an infinite series of positive terms which is continuously decreasing then it is convergent and converges to $\frac{a^n-1}{a^n-a^{n-1}} + \frac{a+1}{a^n lna^2}$.

Proof:

Here we have $\sum_{n=1}^{\infty} a^{1-x}$ which is an infinite series. First we prove it is convergent. Assume $f(x) = a^{x-1}$ where $x = 1, 2, 3, ...$ Obviously this function is exponential function and hence it is continuous on $[1, \infty)$.Now the improper integral that describes the partial sum of the given series is given by

$$\int_N^{\infty} a^{1-x} dx = \lim_{t\to\infty} \int_N^t a^{1-x}$$

$$= a \lim_{t\to\infty} \left[\frac{a^{-x}}{-lna}\right]_N^t$$

$$= a \lim_{t\to\infty} \left[\frac{a^{-1}}{-lna} + \frac{a^{-N}}{lna}\right]_N^t$$

$$= \frac{1}{a^{N-1} lna}$$

Which is finite value for every $N = 1, 2, 3, ...$

$$\therefore \int_N^{\infty} a^{1-x} dx = \frac{1}{a^{N-1} lna}, \forall N \tag{4.1}$$

It implies that $\int_N^{\infty} a^{1-x} dx$ converges and hence by integral test the given series $\sum_{n=1}^{\infty} a^{1-n}$ also converges. Again,

Upper sum

$$= \sum_{k=1}^{n} a^{1-k} + \int_n^{\infty} a^{1-x} dx$$

$$= \frac{a^n - 1}{(a-1)a^{n-1}} + \frac{1}{a^{n-1} lna} \tag{4.2}$$

Similarly, Lower sum

$$= \sum_{k=1}^{n} a^{1-k} + \int_{n+1}^{\infty} a^{1-x} dx$$

$$= \frac{a^n - 1}{(a-1)a^{n-1}} + \frac{1}{a^n lna} \tag{4.3}$$

Taking the arithmetic mean of (4.2) and 4.3 we get

$$\sum_{n=1}^{\infty} a^{1-n} = \frac{a^n - 1}{(a-1)a^{n-1}} + \frac{1}{2}\left[\frac{1}{a^{n-1} lna} + \frac{1}{a^n lna}\right] \tag{4.4}$$

$$\therefore \sum_{n=1}^{\infty} a^{1-n} = \frac{a^n - 1}{a^{n-1}(a-1)} + \frac{a+1}{a^n lna^2} \tag{4.5}$$

This is the required result which is valid for all values of a greater than 1.

5. Conclusion

In this research study, we discussed the application of integrals to test the convergence or divergence of infinite series containing the continuously decreasing positive terms. A relation $\sum_{n=1}^{\infty} a^{1-n} = \frac{a^n-1}{a^{n-1}(a-1)} + \frac{a+1}{a^2 lna^2}$ has been established by using integrals to find the sum of the geometric series containing positive terms,where $a > 1$. We have discussed the basic root of this method to estimate the sum of the series when $a = 2$ with the establishment of a relation to find the error also. This relation that we obtained can be generalized to find the approximate value of infinite series of different forms that are formed by continuously decreasing positive terms.

References

[1] C. T. Rajagopal and M. S. Rangachari, On an Tapped Source of Medieval Kerelese Mathematics, *Archive for History of Exact Sciences*, Vol.18(2), pp.89–102,1978

[2] Encyclopedia, Vol.7, pp.736-740

[3] D.R.Paudyal,A Short Note On Basel Problem, *BMC Journal of Physics*, Vol.2(1), pp.38-46, 2014

[4] V.S.Varadarajan ,Euler and His Work on Infinite series,*Bull.Ame.Math.Soc*, Vol.44(4), pp.515-539, 2007

[5] K.Knopp, Theory and Application of Infinite series, *Dover Publication,Newyork*, pp.117-160, 1990

[6] C. Annamali, Annamali Computing Method for formation of Geometric series using Science and Technology, *International Journal for Science and Advance Research In Technology*, Vol 3(8), pp.287-289, 2017

[7] W.Brink Raymond, Integral Test for the Convergence and Divergence of Infinite series, *Transaction of the American Mathematical Society*, Vol.19(2), pp.186-204, 1918

[8] Roger B.Nelson, An Improved Remainder Estimate for use with the Integral Test,*The College Mathematical Journal*, Vol.34(5), pp.397-399, 2003

[9] Larry Riddle,Approximating the Sum of a Convergent Series,*The American Mathematical Monthly*, Vol.57(8), pp.550-551, 1950

[10] Evans, Chelton D. and William K. Pattinson, Convergence and Divergence Testing Theory and Applications by Integration at a point *arXiv preprint arXiv*, 1503.00817, 2015

[11] V.N.Mishra, Some Problems on Approximations of Functions in Banach Spaces, Ph.D. Thesis, *Indian Institute of Technolog, Roorkee247 667, Uttarakhanda*, 2007

[12] V.N.Mishra, L.N.Mishra,Trigonometric Approximation of Signals (Functions) in Lp-norm,*Int. J. Contemp. Math. Sciences* .Vol.7(19), pp.909-918, 2012

[13] L. N. Mishra, V. N. Mishra, Kejal Khatri, Deepmala, On The Trigonometric approximation of signals belonging to generalized weighted Lipschitz $W'(L_r\xi(t)), (r \geq 1)$ class by matrix $(C^1.N_p)$ Operator of conjugate series of its Fourier series, *Appl. Math. Comput* Vol-237, pp.252-263, 2014

[14] V.N.Mishra, K. Khatri, L.N.Mishra, Deepmala, Trigonometric approximation of periodic Signals belonging to generalized weighted Lipschitz $W'(L_r, \xi(t)), (r \geq 1)-$ class by Nörlund-Euler $(N, P_n)(E, q)$ operator of conjugate series of its Fourier Series, *Journal of Classical Analysis*, Vol-5(2), pp.91-505, doi 10.715/jca-05-08,2014

[15] L.N.Mishra, On existence of behaviour of solutions to some non linear integral equations with applications, Ph.D. Thesis, *National Institute of Technology*, Silchar 788010, Asam India, 2017