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Research Article

Three Dimensional Fractional Mellin Transform, and its Applications

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Abstract

We introduce three dimensional fractional Mellin transform and establish analytic theorem, boundedness theorem, inversion theorem and uniqueness theorem for three dimensional fractional Mellin transform. We present propositions of 3-dimensional fractional Mellin transform. We give some applications of 3-dimensional fractional Mellin transform for solving PDE's.

Keywords: Three dimensional fractional Mellin transform; analytic theorem; boundedness theorem; uniqueness theorem; inversion theorem; Mboctara equation.
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1 Introduction

In modern era, the work of integral transform is very important in the daily life [1- 3]. In the present scenario, the fractional integral transform has been widely used in signal processing, image reconstruction, pattern recognition, and many more fields [4, 5]. Mellin transform is derived from a complex Fourier transform [1] called scale covariant transform [6,7]. Mellin

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transform has many applications, such as algorithms, correlators, navigation, vowel recognition, cryptographic scheme quantum calculus, radar classification of ships, electromagnetic, stress distribution, agriculture, medical stream, statistics, probability, signal processing, optics, and pattern recognition [8, 9]. Akay and Boudreause [7] have introduced the fractional Mellin transform to generalize the scale covariant transform. In connection with fractional order, the fractional Fourier transform was given by Namias [6], which was dependent on continuous parameters. The applications of fractional Mellin transform have been given in controlling the range of rotation and scaling of the signal system [2, 10,], whereas the fractional Fourier transform is only restricted to object recognition in the signal system [8,10-15]. So, the fractional Mellin transform is an essential aspect for analysis of the control system, stability of electrical networks, and many more in Science and Engineering. Sharma and Deshmukh [8] have given applications of two-dimensional fractional Mellin transform. To solve some kind of third-order differential equations is a challenging task [11]. In this context, we present propositions that extend the Mellin transform to a fractional form. We also explore the concept of 3-DFrMT to solve certain third- order homogeneous Mboctara PDE's. This work represents an original scientific contribution and introduces new findings in the field.

2 3-DFrMT

Definition 2.1. 3DFrMT with parameter Λ of $h(z, w, \mho)$ denoted by 3DFrMT $\{h(z, w, \mho)\}$ and it is defined by

$$3DFrMT\{h(z,w,\mho)\} = H_\Lambda(\beta,\gamma,\delta) = \int_0^\infty \int_0^\infty \int_0^\infty h(z,w,\mho)\ K_\Lambda(z,w,\mho,\beta,\gamma,\delta)\ dz\ dw\ d\mho \tag{2.1}$$

where, the kernal

$$K_\Lambda(z,w,\mho,\beta,\gamma,\delta)$$

$$= z^{\frac{2\pi i\beta}{\sin\Lambda}-1} w^{\frac{2\pi i\gamma}{\sin\Lambda}-1} \mho^{\frac{2\pi i\delta}{\sin\Lambda}-1} e^{\pi i\left[\frac{1}{\tan\Lambda}(\beta^2+log^2 z)+\frac{1}{\tan\Lambda}(\gamma^2+log^2 w)+\frac{1}{\tan\Lambda}(\delta^2+log^2\mho)\right]}$$

for, $0 < \Lambda \leq \frac{\pi}{2}$.

Now we give the analytic theorem in 3DFrMT.

Theorem 2.2. Let $h \in E^*(R^n)$ and its FrMT be defined by (5), then $H_\Lambda(\chi, \zeta, \rho)$ is analytic on C^n if the ϖ, τ, κ supp h $\subset S_\varpi$, S_τ *and* S_κ, where

$$S_\varpi = \{z : z \in R^n, |z| \leq \varpi, \varpi > 0\},$$

$$S_\tau = \{w : w \in R^n, |w| \leq \tau, \tau > 0\},$$

$$S_\kappa = \{\mho : \mho \in R^n, |\mho| \leq \kappa, \kappa > 0\}.$$

Moreover $H_\Lambda(\chi, \zeta, \rho)$ is differentiable and $D_\beta^\chi \, D_\gamma^\zeta \, D_\delta^\rho \, H_\Lambda(\beta, \gamma, \delta) = < h(z, w, \mho, \beta, \gamma, \delta) >$
.

Proof. Let

β: $(\beta_1, \beta_2, - - -\beta_j, - - -\beta_n) \in C^n$,

γ: $(\gamma_1, \gamma_2, - - -\gamma_j - - - \gamma_n) \in C^n$,

δ: $(\delta_1, \delta_2, - - -\delta_j, - - -\delta_n) \in C^n$.

We first prove that

$$\frac{\partial}{\partial \beta_j} \frac{\partial}{\partial \gamma_j} \frac{\partial}{\partial \delta_j} H_\Lambda(\beta, \gamma, \delta) = < h(z, w, \mho), \frac{\partial}{\partial \beta_j} \frac{\partial}{\partial \gamma_j} \frac{\partial}{\partial \delta_j} K_\Lambda(z, w, \mho, \beta, \gamma, \delta) >.$$

For fixed $\beta_j \neq 0$, choose two concentric circles C_1, C_2 with centre at σ and radius R_1, R_2 respectively such that $0 < R_1 < R_2 < |\beta_j|$. Let $\Delta\beta_{i,j}$ be a complex increment satisfying

$0 < |\beta_j| < R_1$.

Again for fixed $\gamma_j \neq 0$. Again choose two concentric circles C_3, C_4 with centre at η and radius R_3, R_4 respectively such that $0 < R_3 < R_4 < |\gamma_j|$. Let $\Delta\gamma_{i,j}$ be a complex increment satisfying $0 < |\gamma_j| < R_3$.

Also for fixed $\delta_j \neq 0$, choose two concentric circles C_5, C_6 with centre at ι and radius R_5, R_6 respectively such that $0 < R_5 < R_6 < |\delta_j|$. Let $\Delta\delta_{i,j}$ be a complex increment satisfying $0 < |\Delta\delta_j| < R_5$.

Consider $$\frac{H_\Lambda|(\beta_j+\Delta\beta_j),\gamma_j,\delta_j|-H_\Lambda|\beta_j,\gamma_j,\delta_j|}{\Delta\beta_j}\quad\frac{H_\Lambda|\beta_j,(\gamma_j+\Delta\gamma_j),\delta_j|-H_\Lambda|\beta_j,\gamma_j,\delta_j|}{\Delta\gamma_j}$$
$$\frac{H_\Lambda|\beta_j,\gamma_j,(\delta_j+\Delta\delta_j)|-H_\Lambda|\beta_j,\gamma_j,\delta_j|}{\Delta\delta_j} - < h(z,w,\mho)\ \frac{\partial}{\partial\beta_j}\frac{\partial}{\partial\gamma_j}\frac{\partial}{\partial\delta_j}\ K_\Lambda(z,w,\mho,\beta,\gamma,\delta) >$$
$$= < h(z,w,\mho), \Upsilon_{\Delta\beta_j\Delta\gamma_j\Delta\delta_j}(z,w,\Im) > \tag{2.2}$$

$K_\Lambda(z,w,\mho,\beta_j,\gamma_j,\delta_j)$.

We have,

$$D_z^\chi D_w^\zeta D_\mho^\rho\ K_\Lambda(z,w,\mho,\beta,\gamma,\delta)$$
$$= D_z^\chi D_w^\zeta D_\mho^\rho\left[z^{\frac{2\pi i\beta}{\sin\Lambda}-1}\ w^{\frac{2\pi i\gamma}{\sin\Lambda}-1}\ \mho^{\frac{2\pi i\delta}{\sin\Lambda}-1}\ e^{\pi i\left[\frac{1}{\tan\Lambda}(\beta^2+log^2 z)+\frac{1}{\tan\Lambda}(\gamma^2+log^2 w)+\frac{1}{\tan\Lambda}(\delta^2+log^2\mho)\right]}\right]$$

$$= \sum_{A=0}^{\chi}\sum_{U=0}^{A}\sum_{B=0}^{\zeta}\sum_{R=0}^{B}\sum_{M=0}^{\rho}\sum_{L=0}^{M}\binom{\chi}{A}\binom{\zeta}{B}\binom{\rho}{M}\ E(\beta)\ \Omega(\gamma)\ T(\delta)\ A!\ B!\ M!\ z^{-\chi+U}\ w^{-\zeta+R}\ \mho^{-\rho+L}$$
$$\left(\frac{2\pi i}{\tan\Lambda}\right)^{A-U}\left(\frac{2\pi i}{\tan\Lambda}\right)^{B-R}\left(\frac{2\pi i}{\tan\Lambda}\right)^{M-L}(\log z)^{A-U}\ (\log w)^{B-R}\ (\log\mho)^{M-L}\ \ C_U(z)\ \ C_R(w)$$
$$C_L(\mho)\ K_\Lambda(z,w,\mho,\beta,\gamma,\delta),$$

where, $E(\beta)$, $\Omega(\gamma)$ $\quad and\, T(\delta)$ is a polynomials in β,γ,δ.

Since $z,w,\mho\in R^n$ and fixed χ, ζ, ρ and Λ are ranging from 0 to $\frac{\pi}{2}$,

$D_z^\chi\ D_w^\zeta\ D_\mho^\rho\ K_\Lambda(z,w,\mho,\beta,\gamma,\delta)$ is analytic inside and on C'' , C_1'' and C_2'', we have by Cauchy's integral formula

$$D_z^\chi\ D_w^\zeta\ D_\mho^\rho\,\Upsilon_{\Delta\beta_j\Delta\gamma_j\Delta\delta_j}(z,w,\mho)$$

$$= \frac{1}{8\pi^3 i^3}D_z^\chi D_w^\zeta D_\mho^\rho\int_{C''}\int_{C_1''}\int_{C_2''}K_\Lambda(z,w,\mho,\tilde\beta,\tilde\gamma,\tilde\delta)\left[\frac{1}{\Delta\beta_j}\left(\frac{1}{X-\beta_j-\Delta\beta_j}-\frac{1}{X-\beta_j}\right)-\frac{1}{(X-\beta_j)^2}\right]$$
$$\left[\frac{1}{\Delta\gamma_j}\left(\frac{1}{Y-\gamma_j-\Delta\gamma_j}-\frac{1}{Y-\gamma_j}\right)-\frac{1}{(Y-\gamma_j)^2}\right]\left[\frac{1}{\Delta\delta_j}\left(\frac{1}{Z-\delta_j-\Delta\delta_j}-\frac{1}{Z-\delta_j}\right)-\frac{1}{(Z-\delta_j)^2}\right]dX\ dY\ dZ$$

where,

$\tilde{\beta} = \beta_1, \beta_2, - - -\beta_{j-1}, X, \beta_{j+1}, - - -, \beta_n,$

$\tilde{\gamma} = \gamma_1, \gamma_2, - - -\gamma_{j-1}, Y, \gamma_{j+1}, - - -, \gamma_n,$

$\tilde{\delta} = \delta_1, \delta_2, - - -\delta_{j-1}, Z,\ \delta_{j+1}, - - -\delta_n$

$$= \frac{\Delta\beta_j \Delta\gamma_j \Delta\delta_j}{-8\pi^3 i} \int_{C''} \int_{C_1''} \int_{C_2''} \frac{T(z, w, \mho, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta})}{(X - \beta_j - \Delta\beta_j)(X - \beta_j)^2 (Y - \gamma_j - \Delta\gamma_j)(Y - \gamma_j)^2}$$
$(Z - \delta_j - \Delta\delta_j)(Z - \delta_j)^2$

dX dY dZ

But for all X $\in C''$, $Y \in C_1''$ and $Z \in C_2''$ and z,w and $\mho$ restricted to a compact subset of R^n, $0 < \Lambda \leq \frac{\pi}{2}$, $T(z, w, \mho, \beta, \gamma, \delta) = D_z^{\chi}\, D_w^{\zeta}\, D_{\mho}^{\rho}\, K(z, w, \mho, \beta, \gamma, \delta)$ is bounded by a constant G.

Moreover $|X-\beta_j-\Delta\beta_j| > R_2-R_1 > 0$ and $|X-\beta_j| = R_2$, $|Y-\gamma_j-\Delta\gamma_j| > R_4-R_3 > 0$ and

$|Y - \gamma_j| = R_4$, $|Z - \delta_j - \Delta\delta_j| > R_6 - R_5 > 0$ and $|Z - \delta_j| = R_6$.

Therefore we have,

$$\left| D_z^{\chi}\, D_w^{\zeta}\, D_{\mho}^{\rho}\, \Upsilon_{\Delta\beta_j \Delta\gamma_j \Delta\delta_j}(z, w, \mho) \right|$$
$$\leq \frac{|\Delta\beta_j|\ |\Delta\gamma_j|\ |\quad \Delta\delta_j|}{8\pi^3 i} \int_{c''} \int_{c_1''} \int_{c_2''} \frac{G}{(R_2 - R_1) R_2^2\, (R_4 - R_3) R_4^2 (R_6 - R_5) R_6^2}\ \ |dX|\ \ |dY|\ \ |dZ|$$
$$\leq \frac{|\Delta\beta_j|\ |\Delta\gamma_j||\Delta\delta_j|\ G}{(R_2 - R_1) R_2\ (R_4 - R_3) R_4\ (R_6 - R_5) R_6}$$

Thus, as $|\Delta\beta_j| \to 0$, $|\Delta\gamma_j| \to 0$ and $|\Delta\delta_j| \to 0$, $D_z^{\chi}\ D_w^{\zeta}\ D_{\mho}^{\rho}\ \Upsilon_{\Delta\beta_j \Delta\gamma_j \Delta\delta_j}(z, w, \mho) \to 0$ uniformly on the compact subset of R^n. Thus $\Upsilon_{\beta_j, \gamma_j, \delta_j}(z, w, \mho)$ converges in $E(R^n)$ to zero.

Since h $\in (E^*)$, therefore (6) is $\to 0$. Thus $H_{\Lambda}(\beta, \gamma, \delta)$ is differentiable with respect to β_j, γ_j and δ_j. But this is true for all i, j= 1,2,- - -,n. Therefore, $H_{\Lambda}(\beta, \gamma, \delta)$ is analytic on C^n and

$$D_{\beta}^{\chi}\, D_{\gamma}^{\zeta}\, D_{\delta}^{\rho}\, H_{\Lambda}(\beta, \gamma, \delta) = < h(z, w, \mho, \beta, \gamma, \delta), D_{\beta}^{\chi}\, D_{\gamma}^{\zeta}\, D_{\delta}^{\rho}\ \ K_{\Lambda}(z, w, \mho, \beta, \gamma, \delta) > .$$

Now we give boundedness theorem in 3DFrMT.

Theorem 2.3. Let $h(z,w,\mho) \in E^*(R^n)$ and its FrMT be defined by (2.3). Let supp h $\subset s_\varpi \cup s_\tau \cup s_\kappa$, where $S_\varpi = \{z : z \in R^n, |z| \le \varpi, \varpi > 0\}$, $S_\tau = \{w : w \in R^n, |w| \le \tau, \tau > 0\}$ and

$S_\kappa = \{\mho : \mho \in R^n, |\mho| \le \kappa, \kappa > 0\}$, then for all $\beta, \gamma, \delta \in C^n$ for each $\Xi > 0$, there exist a constant $C_2 > 0$ and a positive integer $\psi, \sigma, \xi \in I^+$ such that for $0 < \Lambda \le \dfrac{\pi}{2}$, where I^+ is the set of positive integer.

$$|H_\Lambda(\beta,\gamma,\delta)| \le C_2 \max_{\substack{\chi\leqslant\psi\\ \zeta\leqslant\sigma\\ \rho\leqslant\xi}} C_\Lambda\, C^\star_\Lambda\, C^{\star\star}_\Lambda \Big[\log(\varpi+\Xi_1)\Big]^{A-U}\Big[\log(\varpi+\Xi_2)\Big]^{B-R}\Big[\log(\varpi+\Xi_3)\Big]^{M-L}$$

$$C_U(\varpi+\Xi_1)\;\; C_R(\tau+\Xi_2)\;\; C_L(\kappa+\Xi_3)\;(\varpi+\Xi_1)^{\frac{\pi}{\sin\Lambda}\left[2(I_A\beta)+cos\Lambda(I_A(log(\varpi+\Xi_1))\right]+U-\chi-1}$$
$$(\tau+\Xi_2)^{\frac{\pi}{\sin\Lambda}\left[2(I_B\gamma)+cos\Lambda(I_B(log(\tau+\Xi_2))\right]+R-\zeta-1}(\kappa+\Xi_3)^{\frac{\pi}{\sin\Lambda}\left[2(I_M\delta)+cos\Lambda(I_M(log(\kappa+\Xi_3))\right]+L-\rho-1}.$$

Proof. By the Boundedness property of the generalized function, there exit a constant C_2 and a positive integer ψ, σ and ξ, such that

$$\begin{aligned}\Big|H_\Lambda(\beta,\gamma,\delta)\Big| &\le \Big| < h(z,w,\mho), K_{\Lambda,\hbar,\Im}(z,w,\mho,\beta,\gamma,\delta) > \Big| \\ &\le C_2 \max_{\substack{\chi\leqslant\psi\\ \zeta\leqslant\sigma\\ \rho\leqslant\xi}} \sup_{\substack{z\in R^n\\ w\in R^n\\ \mho\in R^n}} \Big|D^\chi_z\, D^\zeta_w\, D^\rho_\mho\, K_\Lambda(z,w,\mho,\beta,\gamma,\delta)\Big|\end{aligned}$$

$$\le C_2 \max_{\substack{\chi\leqslant\psi\\ \zeta\leqslant\sigma\\ \rho\leqslant\xi}} \sup_{\substack{z\in R^n\\ w\in R^n\\ \mho\in R^n}} \Bigg| \sum_{A=0}^{\chi}\sum_{U=0}^{A}\sum_{B=0}^{\zeta}\sum_{R=0}^{B}\sum_{M=0}^{\rho}\sum_{L=0}^{M}\binom{\chi}{A}\binom{\zeta}{B}\binom{\rho}{M} v(\beta)\,\Omega(\gamma)\,n(\delta)\,A!\,B!\,M!$$
$$z^{\frac{2\pi i\beta}{\sin\Lambda}-1-\chi+U}\; w^{\frac{2\pi i\gamma}{\sin\Lambda}-1-\zeta+R}\; \mho^{\frac{2\pi i\delta}{\sin\Lambda}-1-\rho+L}\left(\frac{2\pi i}{\tan\Lambda}\right)^{A-U}\left(\frac{2\pi i}{\tan\Lambda}\right)^{B-R}\left(\frac{2\pi i}{\tan\Lambda}\right)^{M-L}\Bigg|$$
$$(\log\, z)^{A-U}\,(\log\, w)^{B-R}\,(\log\,\mho)^{M-L}\, C_U(z)C_R(w)\;\; C_L(\mho)$$
$$e^{\pi i\left[\frac{1}{\tan\Lambda}(\beta^2+log^2 z)+\frac{1}{\tan\Lambda}(\gamma^2+log^2 w)+\frac{1}{\tan\Lambda}(\delta^2+log^2\mho)\right]}$$
$$\le C_2 \max_{\substack{\chi\leqslant\psi\\ \zeta\leqslant\sigma\\ \rho\leqslant\xi}} C_\Lambda\, C^\star_\Lambda\, C^{\star\star}_\Lambda\,(\varpi+\Xi_1)^{\frac{2\pi(I_A\beta)}{\sin\Lambda}+U-\chi-1}(\tau+\Xi_2)^{\frac{2\pi(I_B\gamma)}{\sin\Lambda}+R-\zeta-1}$$
$$(\kappa+\Xi_3)^{\frac{2\pi(I_M\delta)}{\sin\Lambda}+L-\rho-1}[\log(\varpi+\Xi_1)]^{A-U}\;\;[\log(\tau+\Xi_2)]^{B-R}\;\;[\log(\kappa+\Xi_3)]^{M-L}$$
$$C_U(\varpi+\Xi_1)\;\; C_R(\tau+\Xi_2)\;\; C_L(\kappa+\Xi_3)(\varpi+\Xi_1)^{\frac{\pi}{\tan\Lambda}(\varpi+\Xi_1)}(\tau+\Xi_2)^{\frac{\pi}{\tan\Lambda}(\tau+\Xi_2)}(\kappa+\Xi_3)^{\frac{\pi}{\tan\Lambda}(\kappa+\Xi_3)}$$

$$\text{where,}\quad C_\Lambda = \sum_{A=0}^{\chi}\sum_{U=0}^{A}\binom{\chi}{A}\, A!\, R(\beta)\Big(\frac{2\pi i}{\tan\Lambda}\Big)^{A-U} e^{\frac{\pi(I_A\beta^2)}{\sin\Lambda}}$$

$$C^\star_\Lambda = \sum_{B=0}^{\zeta}\sum_{R=0}^{B}\binom{\zeta}{B}\, B!\, \hbar(\gamma)\Big(\frac{2\pi i}{tan\Lambda}\Big)^{B-R} e^{\frac{\pi(I_B\gamma^2)}{sin\Lambda}}$$

$$C^{\star\star}_\Lambda = \sum_{M=0}^{\rho}\sum_{L=0}^{M}\binom{\rho}{M}\, M!\, T(\delta)\Big(\frac{2\pi i}{tan\Lambda}\Big)^{M-L} e^{\frac{\pi(I_M\delta^2)}{sin\Lambda}}$$

$$\leq C_2 \max_{\substack{\chi\leqslant\psi\\ \zeta\leqslant\sigma\\ \rho\leqslant\xi}} C_\Lambda\, C^\star_\Lambda\, C^{\star\star}_\Lambda \Big[log(\varpi+\Xi_1)\Big]^{A-U}\Big[log(\varpi+\Xi_2)\Big]^{B-R}\Big[log(\varpi+\Xi_3)\Big]^{M-L} C_U(\varpi+\Xi_1)$$

$$C_R(\tau+\Xi_2)\quad C_L(\kappa+\Xi_3)\quad (\varpi+\Xi_1)^{\frac{\pi}{sin\Lambda}\Big[2(I_A\beta)+cos\Lambda(I_A(log(\varpi+\Xi_1))\Big]+U-\chi-1}$$

$$(\tau+\Xi_2)^{\frac{\pi}{sin\Lambda}\Big[2(I_B\gamma)+cos\Lambda(I_B(log(\tau+\Xi_2))\Big]+R-\zeta-1}(\kappa+\Xi_3)^{\frac{\pi}{sin\Lambda}\Big[2(I_M\delta)+cos\Lambda(I_M(log(\kappa+\Xi_3))\Big]+L-\rho-1}.$$

Now we give inversion theorem in 3DFrMT.

Theorem 2.4. We can find the function $h(z,w,\mho)$ using the inversion formula of 3DFrMT

$$h(z,w,\mho) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H_\Lambda(\beta,\gamma,\delta)\ \widetilde{K}_\Lambda(z,w,\mho,\beta,\gamma,\delta)\, d\beta\, d\gamma\, d\delta,$$

where,

$$\widetilde{K}_\Lambda(z,w,\mho,\beta,\gamma,\delta)$$

$$= \Big(\frac{-i}{\sin^3\Lambda}\Big)\, z^{\frac{-2\pi i\beta}{\sin\Lambda}} w^{\frac{-2\pi i\gamma}{\sin\Lambda}} \mho^{\frac{-2\pi i\delta}{\sin\Lambda}} e^{-\pi i\Big[\frac{1}{\tan\Lambda}(\beta^2+log^2 z)+\frac{1}{\tan\Lambda}(\gamma^2+log^2 w)+\frac{1}{\tan\Lambda}(\delta^2+log^2\mho)\Big]}.$$

Proof. We have

$$3DFrMT\{h(z,w,\mho)\} = H_\Lambda(\beta,\gamma,\delta) = \int_0^\infty\int_0^\infty\int_0^\infty h(z,w,\mho)\, K_\Lambda(z,w,\mho,\beta,\gamma,\delta)\, dz\, dw\, d\mho$$

$$= \int_0^\infty\int_0^\infty\int_0^\infty h(z,w,\mho)\ z^{\frac{2\pi i\beta}{\sin\Lambda}-1}\ w^{\frac{2\pi i\gamma}{\sin\Lambda}-1}\ \mho^{\frac{2\pi i\delta}{\sin\Lambda}-1}$$

$$e^{\pi i\left[\frac{1}{\tan\Lambda}(\beta^2+log^2 z)+\frac{1}{\tan\Lambda}(\gamma^2+log^2 w)+\frac{1}{\tan\Lambda}(\delta^2+log^2\mho)\right]}\ dz\ dw\ d\mho.$$

$$\therefore \quad e^{\frac{-\pi i\beta^2}{\tan\Lambda}}\ e^{\frac{-\pi i\gamma^2}{\tan\Lambda}}\ e^{\frac{-\pi i\delta^2}{\tan\Lambda}} H_\Lambda(\beta,\gamma,\delta)$$

$$= \int_0^\infty\int_0^\infty\int_0^\infty h(z,w,\mho)\ z^{\frac{2\pi i\beta}{\sin\Lambda}-1}\ w^{\frac{2\pi i\gamma}{\sin\Lambda}-1}\ \mho^{\frac{2\pi i\delta}{sin\Lambda}-1}\ e^{\pi i\left[\frac{log^2 z}{\tan\Lambda}+\frac{log^2 w}{\tan\Lambda}+\frac{log^2\mho}{\tan\Lambda}\right]}\ dz\ dw\ d\mho$$

$$= \int_0^\infty\int_0^\infty\int_0^\infty g(z,w,\mho)\ z^{\frac{2\pi i\beta}{\sin\Lambda}-1}\ w^{\frac{2\pi i\gamma}{\sin\Lambda}-1}\ \mho^{\frac{2\pi i\delta}{\sin\Lambda}-1}\ dz\ dw\ d\mho,$$

where,

$$g(z,w,\mho) = h(z,w,\mho)\ e^{\pi i\left[\frac{log^2 z}{\tan\Lambda}+\frac{log^2 w}{\tan\Lambda}+\frac{log^2\mho}{\tan\Lambda}\right]}$$

$$= M\{g(z,w,\mho)\}\ (\tau)\ (\Omega)\ (\Psi),$$

where, $\tau = \left(\frac{2\pi i\beta}{\sin\Lambda}\right)$, $\Omega = \left(\frac{2\pi i\gamma}{\sin\Lambda}\right)$, $\Psi = \left(\frac{2\pi i\delta}{\sin\Lambda}\right)$.

Using inverse Mellin transform

$$g(z,w,\mho) = \frac{1}{8\pi^3}\int_{-\infty}^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty G(\tau,\Omega,\Psi)\ \ z^{-\tau}\ \ w^{-\Omega}\mho^{-\Psi} d\tau\ d\Omega\ d\Psi,$$

where,

$$G(\tau,\Omega,\Psi) = e^{\frac{-\pi i\beta^2}{tan\Lambda}}\ e^{\frac{-\pi i\gamma^2}{tan\Lambda}}\ e^{\frac{-\pi i\delta^2}{tan\Lambda}} H_\Lambda\left(\frac{\tau sin\Lambda}{2\pi i},\frac{\Omega sin\Lambda}{2\pi i},\frac{\Psi sin\Lambda}{2\pi i}\right)$$

$$\therefore h(z,w,\mho)\ e^{\pi i\left[\frac{log^2 z}{\tan\Lambda}+\frac{log^2 w}{\tan\Lambda}+\frac{log^2\mho}{\tan\Lambda}\right]} = \frac{1}{8\pi^3}\int_{-\infty}^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty G(\tau,\Omega,\Psi)\ z^{-\tau}\ w^{-\Omega}\ \mho^{-\Psi} d\tau\ d\Omega\ d\Psi$$

$$\therefore h(z,w,\mho) = \frac{1}{8\pi^3}\int_{-\infty}^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty e^{\frac{-\pi i\beta^2}{\tan\Lambda}}\ e^{\frac{-\pi i\gamma^2}{\tan\Lambda}}\ e^{\frac{-\pi i\delta^2}{\tan\Lambda}}\ H_{\Lambda,\hbar,\Im}(\beta,\gamma,\delta)\ \ z^{-\frac{2\pi i\beta}{\sin\Lambda}}\ \ w^{-\frac{2\pi i\gamma}{\sin\Lambda}}$$

$$\mho^{\frac{-2\pi i\delta}{\sin\Lambda}} \left(\frac{2\pi i}{\sin\Lambda}\right) \left(\frac{2\pi i}{\sin\Lambda}\right) \left(\frac{2\pi i}{\sin\Lambda}\right) e^{-\pi i\left[\frac{log^2 z}{\tan\Lambda}+\frac{log^2 w}{\tan\Lambda}+\frac{log^2 \mho}{\tan\Lambda}\right]} \, d\beta \;\, d\gamma \;\, d\delta$$

$$h(z,w,\mho) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H_\Lambda(\beta,\gamma,\delta)\; \widetilde{K}_\Lambda(z,w,\mho,\beta,\gamma,\delta)\, d\beta\, d\gamma\, d\delta$$

where,

$$\widetilde{K}_\Lambda(z,w,\mho,\beta,\gamma,\delta)$$

$$= \left(\frac{-i}{\sin^3\Lambda}\right) z^{\frac{-2\pi i\beta}{\sin\Lambda}} w^{\frac{-2\pi i\gamma}{\sin\Lambda}} \mho^{\frac{-2\pi i\delta}{\sin\Lambda}} e^{-\pi i\left[\frac{1}{\tan\Lambda}(\beta^2+log^2 z)+\frac{1}{\tan\Lambda}(\gamma^2+log^2 w)+\frac{1}{\tan\Lambda}(\gamma^2+log^2 \mho)\right]}.$$

Now we give uniqueness theorem in 3DFrMT.

Theorem 2.5. If $H_\Lambda(\beta,\gamma,\delta)$ = 3DFrMT $\{h(z,w,\mho)\}$, $G_\Lambda(\beta,\gamma,\delta)$ = 3DFrMT $\{g(z,w,\mho)\}$,
$Q_\Lambda(\beta,\gamma,\delta)$ =3DFrMT $\{q(z,w,\mho)\}$ for, $0<\Lambda\le\frac{\pi}{2}$ and supp h $\subset S_\varpi$, supp g $\subset S_\tau$, supp q $\subset S_\kappa$, where

$$S_\varpi = \{z : z\in R^n, |z|\le\varpi, \varpi>0\},$$

$$S_\tau = \{w : w\in R^n, |w|\le\tau, \tau>0\},$$

$$S_\kappa = \{\mho : \mho\in R^n, |\mho|\le\kappa, \kappa>0\}.$$

If, $H_\Lambda(\beta,\gamma,\delta) = G_\Lambda(\beta,\gamma,\delta) = Q_\Lambda(\beta,\gamma,\delta)$ then h = g = q in the sense of equality in E^*.

Proof. By inversion theorem

$$\begin{aligned} & h-g-q \\ & = \lim_{M\to\infty}\frac{1}{2\pi}\int_{-M}^{M}\widetilde{K}_\Lambda(z,w,\mho,\beta,\gamma,\delta)\Big[H_\Lambda(\beta,\gamma,\delta)-G_\Lambda(\beta,\gamma,\delta)-Q_\Lambda(\beta,\gamma,\delta)\Big]d\beta\;\, d\gamma\;\, d\delta \\ & = 0. \end{aligned}$$

3 Some Propositions

Example 3.1. Show that

$$\{3DFrMT(1)\}\ (\beta,\gamma,\delta) = (i\ tan\Lambda)^{3/2} e^{-\pi i tan\Lambda(\beta^2+\gamma^2+\delta^2)}$$

Proof. We have $\{3DFrMT(1)\}(\beta,\gamma,\delta)$

$$= \int_0^\infty \int_0^\infty \int_0^\infty (1)\ z^{\frac{2\pi i\beta}{sin\Lambda}-1} w^{\frac{2\pi i\gamma}{sin\Lambda}-1} \mho^{\frac{2\pi i\delta}{sin\Lambda}-1} e^{\frac{\pi i}{tan\Lambda}(\beta^2+\gamma^2+\delta^2+log^2 z+log^2 w+log^2\mho)}\ dz\ dw\ d\mho$$

putting,

$\log$ z $= \varpi \Longrightarrow z = e^{\varpi} \Longrightarrow dz = e^{\omega} d\varpi$

$\log$ w $= \nu \Longrightarrow w = e^{\nu} \Longrightarrow dw = e^{\nu} d\nu$

$\log \mho = \psi \Longrightarrow \mho = e^{\psi} \Longrightarrow d\mho = e^{\psi} d\psi$

$\therefore \{3DFRMT(1)\}\ (\beta,\gamma,\delta)$

$$= \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty (e^{\varpi})^{\frac{2\pi i\beta}{sin\Lambda}-1} (e^{\nu})^{\frac{2\pi i\gamma}{\sin\Lambda}-1} (e^{\Psi})^{\frac{2\pi i\delta}{\sin\Lambda}-1} e^{\frac{\pi i}{\tan\Lambda}(\beta^2+\gamma^2+\delta^2+\varpi^2+\nu^2+\Psi^2)} e^{\varpi} e^{\nu} e^{\Psi}\ d\varpi\ d\nu\ d\Psi$$

where, X $= \frac{2\pi}{sin\Lambda}$, $Y = \frac{\pi}{tan\Lambda}$

$$= e^{\frac{\pi i}{tan\Lambda}(\beta^2+\gamma^2+\delta^2)} \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty e^{iX(\beta\varpi+\gamma\nu+\delta\Psi)}\ e^{iY(\varpi^2+\nu^2+\Psi^2)}\ d\varpi\ d\nu\ d\Psi$$
$$= \left(\frac{\pi i}{Y}\right)^{3/2} e^{i\left\{\frac{\pi}{\tan\Lambda} - \frac{X^2}{4Y}\right\}(\beta^2+\gamma^2+\delta^2)}$$
$$= (i\ \tan\Lambda)^{3/2} e^{-\pi i \tan\Lambda(\beta^2+\gamma^2+\delta^2)}.$$

Example 3.2. Show that

$$\left\{3DFrMT\left(e^{i[\zeta\,(\log m)^2+\xi\,(\log n)^2+\psi\,(log\,l)^2]}\right)\right\}(\beta,\gamma,\delta)$$
$$= \frac{(\pi i)^{3/2}}{\sqrt{XYZ}}\ e^{\frac{\pi i}{\tan\Lambda}(\beta^2+\gamma^2+\delta^2)} e^{\frac{-i\pi^2}{\sin\Lambda\ \cos\Lambda}\left\{\frac{\beta^2}{\pi+\zeta\tan\Lambda}+\frac{\gamma^2}{\pi+\xi\tan\Lambda}+\frac{\delta^2}{\pi+\psi\ \tan\Lambda}\right\}}$$

Proof. We have

$$\left\{3DFrMT\left(e^{i\ [\zeta\log^2 m+\xi\log^2 n+\psi\log^2 l]}\right)\right\}(\beta,\gamma,\delta)$$

$$=\int_0^\infty\int_0^\infty\int_0^\infty \left[e^{i\ [\zeta\ \log^2 m+\xi\ \log^2 n+\psi\ \log^2 l]}\right] m^{\frac{2\pi i\beta}{\sin\Lambda}-1} n^{\frac{2\pi i\gamma}{\sin\Lambda}-1} l^{\frac{2\pi i\delta}{\sin\Lambda}-1}$$

$$e^{\frac{\pi i}{\tan\Lambda}(\beta^2+\gamma^2+\delta^2+\log^2 m+\log^2 n+\log^2 l)}\ \mathrm{dm}\quad \mathrm{dn}\quad \mathrm{dl}$$

$$=\int_0^\infty\int_0^\infty\int_0^\infty m^{\frac{2\pi i\beta}{sin\Lambda}-1} n^{\frac{2\pi i\gamma}{sin\Lambda}-1} l^{\frac{2\pi i\delta}{sin\Lambda}-1} e^{\frac{\pi i}{tan\Lambda}(\beta^2+\gamma^2+\delta^2)} e^{i(\zeta\ log^2 m+\xi\ log^2 n+\psi\ log^2 l)}$$

$$e^{\frac{\pi i}{tan\Lambda}(log^2 m+log^2 n+log^2 l)}\ dm\ dn\ dl$$

Putting,

log m = A $\Rightarrow m=e^A\Rightarrow$ dm = e^A dA

log n = B $\Rightarrow n=e^B\Rightarrow$ dn = e^B dB

log l = C $\Rightarrow l=e^c\Rightarrow$ dl = e^c dC

$$\therefore\left\{3DFrMT\left(e^{i[\zeta\log^2 m+\xi\log^2 n+\psi\log^2 l]}\right)\right\}(\beta,\gamma,\delta)$$

$$=e^{\frac{\pi i}{tan\Lambda}(\beta^2+\gamma^2+\delta^2)}\int_{-\infty}^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty (e^A)^{\frac{2\pi i\beta}{sin\Lambda}}(e^B)^{\frac{2\pi i\gamma}{sin\Lambda}}(e^C)^{\frac{2\pi i\delta}{sin\Lambda}}$$

$$e^{i\left[\left(\zeta+\frac{\pi}{tan\Lambda}\right)A^2+\left(\xi+\frac{\pi}{tan\Lambda}\right)B^2+\left(\psi+\frac{\pi}{tan\Lambda}\right)C^2\right]}\ dA\ dB\ dC$$

$$=e^{\frac{\pi i}{\tan\Lambda}(\beta^2+\gamma^2+\delta^2)}\int_{-\infty}^\infty e^{iGA+iXA^2}dA\int_{-\infty}^\infty e^{iHB+iYB^2}dB\int_{-\infty}^\infty e^{iLC+iZC^2}dC$$

where,

$$G=\frac{2\pi\beta}{\sin\Lambda},\ H=\frac{2\pi\gamma}{\sin\Lambda},\ L=\frac{2\pi\delta}{\sin\Lambda},\ X=\zeta+\frac{\pi}{\tan\Lambda},Y=\xi+\frac{\pi}{\tan\Lambda},\ Z=\psi+\frac{\pi}{\tan\Lambda}$$

$$\therefore\left\{3DFrMT\left(e^{i[\zeta\ \log^2 m+\xi\ \log^2 n+\psi\ \log^2 l]}\right)\right\}(\beta,\gamma,\delta)$$

$$= e^{\frac{\pi i}{\tan\Lambda}(\beta^2+\gamma^2+\delta^2)} \int_{-\infty}^{\infty} e^{iGA+iXA^2} dA \int_{-\infty}^{\infty} e^{iHB+iYB^2} dB \int_{-\infty}^{\infty} e^{iLA+iZA^2} dC$$

$$= \frac{(\pi i)^{3/2}}{\sqrt{XYZ}} e^{\frac{\pi i}{\tan\Lambda}(\beta^2+\gamma^2+\delta^2)} e^{\frac{-i\pi^2}{sin^2\Lambda}\tan\Lambda\left\{\frac{\beta^2}{\pi+\zeta\tan\Lambda}+\frac{\gamma^2}{\pi+\xi\tan\Lambda}+\frac{\delta^2}{\pi+\psi\tan\Lambda}\right\}}$$

$$\therefore \left\{3DFrMT\left(e^{i[\zeta\ (logm)^2+\xi\ (logn)^2+\psi\ (logl)^2]}\right)\right\}(\beta,\gamma,\delta)$$

$$= \frac{(\pi i)^{3/2}}{\sqrt{XYZ}}\ e^{\frac{\pi i}{\tan\Lambda}(\beta^2+\gamma^2+\delta^2)} e^{\frac{-i\pi^2}{\sin\Lambda\cos\Lambda}\left\{\frac{\beta^2}{\pi+\zeta\tan\Lambda}+\frac{\gamma^2}{\pi+\xi\tan\Lambda}+\frac{\delta^2}{\pi+\psi\tan\Lambda}\right\}}.$$

4 Applications

Application 4.1. Let us Consider the Mboctara PDE's

$$\partial_{xyt}\hbar(x,y,t)+\hbar(x,y,t)=0. \tag{4.1}$$

Using the proposed transform, we have

$$(-1)\ \left(\frac{2\pi i\beta}{\sin\Lambda}-1\right)H_\Lambda(\beta-1)\ (-1)\ \left(\frac{2\pi i\gamma}{\sin\Lambda}-1\right)H_\Lambda(\gamma-1)\ (-1)\ \left(\frac{2\pi i\delta}{\sin\Lambda}-1\right)H_\Lambda(\delta-1)$$

$$+\ \hbar(z,w,\mho)=0$$

$$\text{or, } \hbar(z,w,\mho)=\left(\frac{2\pi i\beta}{\sin\Lambda}-1\right)H_\Lambda\left(\frac{2\pi i\gamma}{\sin\Lambda}-1\right)H_\Lambda\left(\frac{2\pi i\delta}{\sin\Lambda}-1\right)\Big[H_\Lambda(\beta-1),H_\Lambda(\gamma-1)H_\Lambda(\delta-1)\Big]$$

$$\hbar(x,y,t)=-FrMT^{-1}\left\{\Big[\left(\frac{2\pi i\beta}{\sin\Lambda}-1\right)\left(\frac{2\pi i\gamma}{\sin\Lambda}-1\right)\left(\frac{2\pi i\delta}{\sin\Lambda}-1\right)\Big]\right.$$
$$\left.\Big[H_\Lambda(\beta-1),H_\Lambda(\gamma-1),H_\Lambda(\delta-1)\Big]\right\}.$$

Application 4.2. Let us consider the Mboctara PDE's

$$\partial_{xyt}\hbar(x,y,t)+\hbar(x,y,t)=1. \tag{4.2}$$

Using the proposed transform, we have

$$(-1)\left(\frac{2\pi i\beta}{\sin\Lambda}-1\right)H_\Lambda(\beta-1)\;(-1)\left(\frac{2\pi i\gamma}{\sin\Lambda}-1\right)H_\Lambda(\gamma-1)\;(-1)\left(\frac{2\pi i\delta}{\sin\Lambda}-1\right)H_\Lambda(\delta-1)$$

$$+\ \hbar(z,w,\mho)=(i\,tan\Lambda)^{3/2}e^{-\pi i\,tan\Lambda(\beta^2+\gamma^2+\delta^2)}$$

$$\text{or, }\hbar(z,w,\mho)=\frac{(i\,tan\Lambda)^{3/2}e^{-\pi i\,tan\Lambda(\beta^2+\gamma^2+\delta^2)}}{\left[\left(\frac{2\pi i\beta}{\sin\Lambda}-1\right)\left(\frac{2\pi i\gamma}{\sin\Lambda}-1\right)\left(\frac{2\pi i\delta}{\sin\Lambda}-1\right)\right]\left[H_\Lambda(\beta-1),H_\Lambda(\gamma-1),H_\Lambda(\delta-1)\right]}$$

$$\hbar(x,y,t)=FrMT^{-1}\left\{\frac{(i\,tan\Lambda)^{3/2}e^{-\pi i\,tan\Lambda(\beta^2+\gamma^2+\delta^2)}}{\left[\left(\frac{2\pi i\beta}{\sin\Lambda}-1\right)\left(\frac{2\pi i\gamma}{\sin\Lambda}-1\right)\left(\frac{2\pi i\delta}{\sin\Lambda}-1\right)\right]\left[H_\Lambda(\beta-1),H_\Lambda(\gamma-1),H_\Lambda(\delta-1)\right]}\right\}.$$

Discussion

The Mellin transform is applicable in different branches of science and engineering for analyzing the algorithms of any system because it is a scale co-variant transform. Since fractional Mellion transform is a generalization of Mellin transform, it is applicable to control the range of rotation and scaling of any signal system. Three dimensional fractional Mellin transform can split the signal into fractional form signals, so its analysis can be more accurate and comfortable than a compact one in the extraction and separation of the image of any system. The present work generalizes the current state of knowledge in our topic because this work is wrathful in more general as in three-dimensional scenarios. The work of three-dimensional fractional Mellin transform and its applications is a novel work in the field of integral transform due to it aids generalized tools with three dimensional to solve more complicated problems. The results represent pioneer importance previously untouched in the field of extension fractionally for the Mellin transform.

Conclusion

Finally, in this paper, we have generalized the three dimensional fractional Mellin transform in the distributional sense and proved its analytic theorem, boundedness theorem, inversion theorem, and uniqueness theorem. Also, we have given propositions of 3-DFrMT, which can also be applied with advantages to solve the different types of problems in any signal processing system, especially in a navigational system as a co-relator to control moments in any specific three-dimensional space.

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