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Research Article

Integration of Hypergeometric Superhyperbolic Functions Using Generalized Hypergeometric Function and Its Applications

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Abstract

In this article , we explore the properties and relations of the hypergeometric superhyberbolic functions in terms of generalized hypergeometric functions. Then we express the hypergeometric superhyberbolic functions with the help of hypergeometric functions to obtain an integral representation related to classical results within the theory of generalized hypergeometric functions. The work emphasises conditions of convergence, and special cases where the results can be expressed in a simple form and provide a unified framework for evaluating integrals involving hypergeometric functions.

Keywords: Pochhammer symbol, Gamma function, Hypergeometric superhyperbolic function.
AMS(MOS) Subject Classification: 33C20.

1 Introduction and Preliminaries

Hypergeometric functions are a class of special functions and are solutions of second-order linear differential equations.The integration of hypergeometric superhyperbolic functions represents an advanced area of mathematical analysis. It has significant applications in mathematical physics,engineering, and computational mathematics. Hypergeometric functions provides a unifying framework for various classes of special functions, including hypergeometric superhyperbolic functions.[3, 4, 10, 12, 13]

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This study focuses on evaluating the integrals of hypergeometric superhyperbolic functions based on well-established theory of generalized hypergeometric functions.The approach enables a systematic way to derive integral representations of hypergeometric superhyperbolic functions and clarify the conditions for their convergence.[6, 7, 9]
The foundation of hypergeometric functions comes from the works of Euler, Gauss, and Riemann in the 18th and 19th centuries. In the 19th century, Clausen defined the generalized hypergeometric function [1, 13]. The hypergeometric superhyperbolic functions are those that arise from the extension of the hyperbolic and hypergeometric functions. The integration technique has evolved to include those of the hypergeometric functions; indeed starting from the pioneer work by Kummer,Whipple, Saalschutz and modern applications of such functions to complex integrals and the solutions of differential equations. [1, 11, 12, 13, 14]

Before proceeding with the main work, we shall explain some basic notations and definitions that are used in this paper.
The gamma function of n is denoted by $\Gamma(n)$ and is defined by [8]

$$\Gamma(n) = \int_0^\infty e^{-t} t^{n-1} dt, \quad Re(n) > 0$$

where $\Gamma(n+1) = n\Gamma(n)$,$\Gamma(n+1) = n!$, & $\Gamma(1/2) = \sqrt{\pi}$.

Beta function of m and n is denoted by $B(m,n)$ and is defined by [6]

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx, \;\; Re(m) > 0, \;\; Re(n) > 0 \;\; \& \;\; B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

Pochhammer Symbol was introduced by the German mathematician Leo Pochhammer (1841-1920) [10]. It is defined by

$$(b)_n = \prod_{k=1}^{n} (b+k-1), \;\; (b)_n = \frac{\Gamma(b+n)}{\Gamma(b)}, \;\; (b)_0 = 1, \;\; (1)_n = n!$$

where n is a non- negative integer.
In 1812, Gauss systematically studied the series

$$1 + \frac{\alpha\beta}{\gamma}\frac{x}{1!} + \frac{\alpha(\alpha+1).\beta(\beta+1)}{\gamma(\gamma+1)}\frac{x^2}{2!} + ... \tag{1.1}$$

The ordinary hypergeometric series is another name for the series (1.1), which is frequently referred to as Gauss's series or function. It can be regarded as an extension of the geometric series.

$$1 + x + x^2 + x^3 + ...$$

The series (1.1) is denoted by ${}_2F_1[\alpha, \beta; \gamma; x]$ or ${}_2F_1\left[\begin{matrix}\alpha, & \beta \\ & & ; x \\ \gamma & \end{matrix}\right]$ and written in the form

$$ {}_2F_1\left[\begin{matrix}\alpha, & \beta & \\ & & ; x \\ \gamma & & \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n} \frac{x^n}{n!} \tag{1.2} $$

where α and β are numerator parameters, while γ is the denominator parameter. For $\gamma \neq 0, -1, -2, -3, \ldots$ and α or β is a negative integer, the series (1.2) will terminate. The Gauss's hypergeometric series (1.2) is

i. convergent if $|x| < 1$, divergent if $|x| > 1$,

ii. convergent if $R(\gamma - \alpha - \beta) > 0$ when $x = 1$,

iii. convergent absolutely if $R(\gamma - \alpha - \beta) > 0$ when $x = -1$,

iv. convergent but not absolutely if $-1 \leq R(\gamma - \alpha - \beta) < 0$ when $x = -1$,

The natural generalization of the Gauss's hypergeometric function ${}_2F_1$ is called the generalized hypergeometric function denoted by ${}_pF_q$ [1, 12] and is defined by

$$ {}_pF_q\left[\begin{matrix}\alpha_1, & ..., & \alpha_p & \\ & & & ; x \\ \beta_1, & ..., & \beta_q & \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n ... (\alpha_p)_n}{(\beta_1)_n ... (\beta_q)_n} \frac{x^n}{n!} \tag{1.3} $$

The generalized hypergeometric function (1.3) converges for all finite x if $p \leq q$. Moreover, it is also convergent for $|x| < 1$ if $p = q + 1$, and absolutely convergent on $|x| > 1$ if $p = q + 1$ and $Re(\sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i) > 0$ [12].

2 The Hypergeometric Superhyperbolic Functions via Generalized Hypergeometric Functions

The family of the hypergeometric functions containing the hypergeometric superhyperbolic sine, hypergeometric superhyperbolic cosine, hypergeometric superhyperbolic tangent, hypergeometric superhyperbolic cotangent, hypergeometric superhyperbolic secant and hypergeometric superhyperbolic cosecant is called the hypergeometric superhyperbolic functions via generalized hypergeometric functions [13, 14].

The hypergeometric superhyperbolic sine and cosine function via generalized hypergeometric function is defined as [13, 14]

$$ {}_pSupersinh_q\left[\begin{matrix} r_1, ..., r_p \\ s_1, ..., s_q \end{matrix} ; x\right] = \sum_{n=0}^{\infty} \frac{(r_1)_{2n+1} ... (r_p)_{2n+1}}{(s_1)_{2n+1} ... (s_q)_{2n+1}} \frac{x^{2n+1}}{(2n+1)!} $$

$$ {}_pSupercosh_q\left[\begin{array}{c} r_1, ..., r_p \\ s_1, ..., s_q \end{array}; x\right] = \sum_{n=0}^{\infty} \frac{(r_1)_{2n}...(r_p)_{2n}}{(s_1)_{2n}...(s_q)_{2n}} \frac{x^{2n}}{(2n)!} $$

where $r_n, s_n \in \mathbb{C}$ and $n, p, q \in \mathbb{N}_0$.

Both hypergeometric superhyperbolic sine and hypergeometric superhyperbolic cosine functions via generalized hypergeometric functions are convergent for all finite x if $p < q$. Moreover these functions are convergent for $|x| < 1$ if $p = q+1$, and absolutely convergent on $|x| > 1$ if $p = q+1$ and $Re(\sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i) > 0$ [13, 14].

Also,

$$ {}_pSupertanh_q = \frac{{}_pSupersinh_q}{{}_pSupercosh_q} $$

Similarly, others ${}_pSupercoth_q$, ${}_pSupersech_q$ and ${}_pSupercosech_q$ can also be defined.

Relation between hypergeometric superhyperbolic functions and generalized hypergeometric functions [13, 14]

$$ i.\ {}_pSupersinh_q\left[\begin{array}{c} r_1, ..., r_p \\ s_1, ..., s_q \end{array}; x\right] = \frac{1}{2}\left[{}_pF_q\left[\begin{array}{c} r_1, ..., r_p \\ s_1, ..., s_q \end{array}; x\right] - {}_pF_q\left[\begin{array}{c} r_1, ..., r_p \\ s_1, ..., s_q \end{array}; -x\right]\right] \tag{2.1} $$

$$ ii.\ {}_pSupercosh_q\left[\begin{array}{c} r_1, ..., r_p \\ s_1, ..., s_q \end{array}; x\right] = \frac{1}{2}\left[{}_pF_q\left[\begin{array}{c} r_1, ..., r_p \\ s_1, ..., s_q \end{array}; x\right] + {}_pF_q\left[\begin{array}{c} r_1, ..., r_p \\ s_1, ..., s_q \end{array}; -x\right]\right] \tag{2.2} $$

Proof:

Let right hand side of (2.1) be denoted by I. Then

$$ I = \frac{1}{2}\left[{}_pF_q\left[\begin{array}{c} r_1, ..., r_p \\ s_1, ..., s_q \end{array}; x\right] - {}_pF_q\left[\begin{array}{c} r_1, ..., r_p \\ s_1, ..., s_q \end{array}; -x\right]\right] $$

$$ = \frac{1}{2}\left[\sum_{n=0}^{\infty} \frac{(r_1)_n...(r_p)_n}{(s_1)_n...(s_q)_n} \frac{x^n}{(n)!} - \sum_{n=0}^{\infty} \frac{(r_1)_n...(r_p)_n}{(s_1)_n...(s_q)_n} \frac{(-x)^n}{(n)!}\right] $$

$$ = \frac{1}{2}\sum_{n=0}^{\infty} \frac{(r_1)_n...(r_p)_n}{(s_1)_n...(s_q)_n} \frac{x^n}{(n)!} \left[1 - (-1)^n\right] $$

where n is an odd positive integer and then replacing n by $2n+1$ we have

$$ I = \sum_{n=0}^{\infty} \frac{(r_1)_{2n+1}...(r_p)_{2n+1}}{(s_1)_{2n+1}...(s_q)_{2n+1}} \frac{x^{2n+1}}{(2n+1)!} = {}_pSupersinh_q\left[\begin{array}{c} r_1, ..., r_p \\ s_1, ..., s_q \end{array}; x\right]. $$

Similarly, the same process can be applied for (2.2)

3 Integral Theorems Involving Generalized Hypergeometric Functions

Theorem 3.1 [1, 12] If $p \leq q+1, Re(r_1) > 0, ..., Re(r_p) > 0, Re(s_1) > 0, ..., Re(s_p) > 0$ and $|x| < 1$, then

$$ {}_pF_q \left[\begin{matrix} r_1, ..., r_p \\ s_1, ..., s_q \end{matrix} ; x \right] = \frac{\Gamma(s_1)}{\Gamma(r_1)\Gamma(s_1 - r_1)} \int_0^1 t^{r_1-1}(1-t)^{s_1-r_1-1} \quad {}_{p-1}F_{q-1} \left[\begin{matrix} r_2, ..., r_p \\ s_2, ..., s_q \end{matrix} ; xt \right] dt \tag{3.1} $$

Corollary(3.1) [2, 12] If $Re(r_3) > Re(r_2) > 0$ and $|x| < 1$, then

$$ {}_2F_1 \left[\begin{matrix} r_1, r_2 \\ r_3 \end{matrix} ; x \right] = \frac{\Gamma(r_3)}{\Gamma(r_2)\Gamma(r_3 - r_2)} \int_0^1 t^{r_2-1}(1-t)^{r_3-r_2-1} (1-tx)^{-r_1} dt $$

$$ = \frac{\Gamma(r_3)}{\Gamma(r_2)\Gamma(r_3 - r_2)} \int_0^1 t^{r_2-1}(1-t)^{r_3-r_2-1} {}_1F_0 \left[\begin{matrix} r_1, \\ - \end{matrix} ; xt \right] \tag{3.2} $$

Corollory (3.2) [2, 12] If $Re(r_2) > Re(r_1) > 0$ and $|x| < 1$, then

$$ {}_1F_1 \left[\begin{matrix} r_1 \\ r_2 \end{matrix} ; x \right] = \frac{\Gamma(r_2)}{\Gamma(r_1)\Gamma(r_2 - r_1)} \int_0^1 t^{r_2-1}(1-t)^{r_3-r_2-1} e^{xt} dt \tag{3.3} $$

Theorem 3.2 [1, 12]

If $Re(\alpha) > 0, Re(\beta) > 0, Re(r_1) > 0, ..., Re(r_p) > 0, Re(s_1) > 0, ..., Re(s_q) > 0, Re(\sum_{k=1}^{q} s_k - \sum_{k=1}^{p} r_k) > 0, |x| < 1$ and $k \in \mathbb{N}$

$$ i. \int_0^t x^{\alpha-1}(t-x)^{\beta-1} \, {}_pF_q \left[\begin{matrix} r_1, ..., r_p \\ s_1, ..., s_q \end{matrix} ; \lambda x^k \right] dx $$

$$ = B(\alpha, \beta) \quad t^{\alpha+\beta-1} \, {}_{p+k}F_{q+k} \left[\begin{matrix} r_1, ..., r_p, \frac{\alpha}{k}, ..., \frac{\alpha+k-1}{k} \\ s_1, ..., s_q, \frac{\alpha+\beta}{k}, ..., \frac{\alpha+\beta+k-1}{k} \end{matrix} ; \lambda t^k \right] \tag{3.4} $$

$$ ii. \int_0^t x^{\alpha-1}(t-x)^{\beta-1} \, {}_pF_q \left[\begin{matrix} r_1, ..., r_p \\ s_1, ..., s_q \end{matrix} ; \lambda x^k (t-x)^s \right] dx $$

$$ = B(\alpha, \beta) t^{\alpha+\beta-1} {}_{p+k}F_{q+k} \left[\begin{matrix} r_1, ..., r_p, \frac{\alpha}{k}, ..., \frac{\alpha+k-1}{k}, \frac{\beta}{s}, ..., \frac{\beta+s-1}{s} \\ s_1, ..., s_q, \frac{\alpha+\beta}{k+s}, ..., \frac{\alpha+\beta+k+s-1}{k+s} \end{matrix} ; \frac{k^k s^s \lambda t^{k+s}}{k+s} \right] \tag{3.5} $$

where λ is a constant.

Theorem (3.3): [9] If $Re(\alpha) > 0, Re(\beta) > 0, Re(r_1) > 0, ..., Re(r_p) > 0, Re(s_1) > 0, ..., Re(s_q) > 0, Re(\sum_{k=1}^{q} s_k - \sum_{k=1}^{p} r_k) > 0$ then

$$ \int_0^t x^{\alpha-1}(t-x)^{\beta-1} \, {}_pF_q \left[\begin{matrix} r_1, ..., r_p \\ s_1, ..., s_q \end{matrix} ; x) \right] dx = B(\alpha, \beta)_{p+k}F_{q+k} \left[\begin{matrix} r_1, ..., r_p, \alpha \\ s_1, ..., s_q, \alpha+\beta \end{matrix} ; 1 \right] \tag{3.6} $$

4 Main Results

In this section ,we evaluate some integrals involving hypergeometric superhyperbolic functions.

Theorem 4.1 If $p \leq q+1, Re(r_1) > 0, ..., Re(r_p) > 0, Re(s_1) > 0, ..., Re(s_p) > 0$ and $|x| < 1$, then

$$i)\ {}_pSupersinh_q\left[\begin{matrix} r_1, ..., r_p \\ s_1, ..., s_q \end{matrix}; x\right] = \frac{\Gamma(s_1)}{\Gamma(r_1)\Gamma(s_1-r_1)}\int_0^1 t^{r_1-1}(1-t)^{s_1-r_1-1}\,{}_{p-1}Supersinh_{q-1}\left[\begin{matrix} r_2, ..., r_p \\ s_2, ..., s_q \end{matrix}; xt\right] dt \tag{4.1}$$

$$ii.\ {}_pSupercosh_q\left[\begin{matrix} r_1, ..., r_p \\ s_1, ..., s_q \end{matrix}; x\right] = \frac{\Gamma(s_1)}{\Gamma(r_1)\Gamma(s_1-r_1)}\int_0^1 t^{r_1-1}(1-t)^{s_1-r_1-1}\,{}_{p-1}Supercosh_{q-1}\left[\begin{matrix} r_2, ..., r_p \\ s_2, ..., s_q \end{matrix}; xt\right] dt \tag{4.2}$$

Proof (i): Let left side of (4.1) be denoted by I and using (2.1) and (3.1), then

$$I = {}_pSupersinh_q\left[\begin{matrix} r_1, ..., r_p \\ s_1, ..., s_q \end{matrix}; x\right] = \frac{1}{2}\left\{{}_pF_q\left[\begin{matrix} r_1, ..., r_p \\ s_1, ..., s_q \end{matrix}; x\right] - {}_pF_q\left[\begin{matrix} r_1, ..., r_p \\ s_1, ..., s_q \end{matrix}; -x\right]\right\}$$

$$= \frac{\Gamma(s_1)}{\Gamma(r_1)\Gamma(s_1-r_1)}\int_0^1 t^{r_1-1}(1-t)^{s_1-r_1-1}\,{}_{p-1}F_{q-1}\left[\begin{matrix} r_2, ..., r_p \\ s_2, ..., s_q \end{matrix}; xt\right] dt$$
$$-\frac{\Gamma(s_1)}{\Gamma(r_1)\Gamma(s_1-r_1)}\int_0^1 t^{r_1-1}(1-t)^{s_1-r_1-1}\,{}_{p-1}F_{q-1}\left[\begin{matrix} r_2, ..., r_p \\ s_2, ..., s_q \end{matrix}; -xt\right] dt$$

$$= \frac{\Gamma(s_1)}{\Gamma(r_1)\Gamma(s_1-r_1)}\int_0^1 t^{r_1-1}(1-t)^{s_1-r_1-1}\,\frac{1}{2}\left[{}_{p-1}F_{q-1}\left[\begin{matrix} r_2, ..., r_p \\ s_2, ..., s_q \end{matrix}; xt\right] - {}_{p-1}F_{q-1}\left[\begin{matrix} r_2, ..., r_p \\ s_2, ..., s_q \end{matrix}; -xt\right]\right] dt$$

$$= \frac{\Gamma(s_1)}{\Gamma(r_1)\Gamma(s_1-r_1)}\int_0^1 t^{r_1-1}(1-t)^{s_1-r_1-1}\,{}_{p-1}Supersinh_{q-1}\left[\begin{matrix} r_2, ..., r_p \\ s_2, ..., s_q \end{matrix}; xt\right] dt$$

Similarly, we can prove (ii).

Corollary 4.1

If $Re(r_3) > Re(r_2) > 0$ and $|x| < 1$,then

$$i.\ {}_2Supersinh_1\left[\begin{matrix} r_1, r_2 \\ r_3 \end{matrix}; x\right] = \frac{\Gamma(r_3)}{\Gamma(r_2)\Gamma(r_3-r_2)}\int_0^1 t^{r_2-1}(1-t)^{r_3-r_2-1}\,{}_1Supersinh_0\left[\begin{matrix} r_1 \\ - \end{matrix}; xt\right] dt \tag{4.3}$$

$$ii.\ {}_2Supercosh_1\left[\begin{matrix} r_1, r_2 \\ r_3 \end{matrix}\ ;x\right] = \frac{\Gamma(r_3)}{\Gamma(r_2)\Gamma(r_3-r_2)}\int_0^1 t^{r_2-1}(1-t)^{r_3-r_2-1}\ {}_1Supercosh_0\left[\begin{matrix} r_1 \\ - \end{matrix}\ ;xt\right]dt \tag{4.4}$$

Corollary 4.2 If $Re(r_2) > Re(r_1) > 0$ where $r_1, r_2, x \in \mathbf{C}$, then

$$i)\ {}_1Supersinh_1\left[\begin{matrix} r_1 \\ r_2 \end{matrix}\ ;x\right] = \frac{\Gamma(r_2)}{\Gamma(r_1)\Gamma(r_2-r_1)}\int_0^1 t^{r_1-1}(1-t)^{r_2-r_1-1}\ sinhxt\ dt \tag{4.5}$$

$$ii)\ {}_1Supercosh_1\left[\begin{matrix} r_1 \\ r_2 \end{matrix}\ ;x\right] = \frac{\Gamma(r_2)}{\Gamma(r_1)\Gamma(r_2-r_1)}\int_0^1 t^{r_1-1}(1-t)^{r_2-r_1-1}\ coshxt\ dt \tag{4.6}$$

Theorem 4.2:

If $Re(\alpha) > 0, Re(\beta) > 0, Re(r_1) > 0, ..., Re(r_p) > 0, Re(s_1) > 0, ..., Re(s_q) > 0, Re(\sum_{k=1}^{q} s_k - \sum_{k=1}^{p} r_k) > 0, |t| < 1$ and $k \in \mathbb{N}$ then

$$i. \int_0^t x^{\alpha-1}(t-x)^{\beta-1}\ {}_pSupersinh_q\left[\begin{matrix} r_1, ..., r_p \\ s_1, ..., s_q \end{matrix}\ ;\lambda x^k\right]dx$$

$$= B(\alpha,\beta)\cdot t^{\alpha+\beta-1}\cdot_{p+k} Supersinh_{q+k}\left[\begin{matrix} r_1, ..., r_p, \frac{\alpha}{k}, ..., \frac{\alpha+k-1}{k} \\ s_1, ..., s_q, \frac{\alpha+\beta}{k}, ..., \frac{\alpha+\beta+k-1}{k} \end{matrix}\ ;\lambda t^k\right] \tag{4.7}$$

$$ii. \int_0^t x^{\alpha-1}(t-x)^{\beta-1}\ {}_pSupercosh_q\left[\begin{matrix} r_1, ..., r_p \\ s_1, ..., s_q \end{matrix}\ ;\lambda x^k\right]dx$$

$$= B(\alpha,\beta)\cdot t^{\alpha+\beta-1}\cdot_{p+k} Supercosh_{q+k}\left[\begin{matrix} r_1, ..., r_p, \frac{\alpha}{k}, ..., \frac{\alpha+k-1}{k} \\ s_1, ..., s_q, \frac{\alpha+\beta}{k}, ..., \frac{\alpha+\beta+k-1}{k} \end{matrix}\ ;\lambda t^k\right] \tag{4.8}$$

where λ is a constant.

Theorem 4.3: If $Re(\alpha) > 0, Re(\beta) > 0, Re(r_1) > 0, ..., Re(r_p) > 0, Re(s_1) > 0, ..., Re(s_q) > 0, p, q \in \mathbf{N_0}, |x| < 1$ and $k, s \in \mathbb{N}$ then

$$i. \int_0^t x^{\alpha-1}(t-x)^{\beta-1}\ {}_pSupersinh_q\left[\begin{matrix} r_1, ..., r_p \\ s_1, ..., s_q \end{matrix}\ ;\lambda x^k (t-x)^s\right]dt$$

$$= B(\alpha,\beta)\cdot t^{\alpha+\beta-1}\cdot_{p+k} Supersinh_{q+k}\left[\begin{matrix} r_1, ..., r_p, \frac{\alpha}{k}, ..., \frac{\alpha+k-1}{k}, \frac{\beta}{s}, ..., \frac{\beta+s-1}{s} \\ s_1, ..., s_q, \frac{\alpha+\beta}{k+s}, ..., \frac{\alpha+\beta+k+s-1}{k+s} \end{matrix}\ ;\frac{k^k s^s \lambda t^{k+s}}{k+s}\right] \tag{4.9}$$

$$ii. \int_0^t x^{\alpha-1}(t-x)^{\beta-1} \ {}_pSupercosh_q \left[\begin{array}{c} r_1,...,r_p \\ s_1,...,s_q \end{array} ; \lambda x^k (t-x)^s \right] dt$$

$$= B(\alpha,\beta)\cdot t^{\alpha+\beta-1} \cdot_{p+k} Supercosh_{q+k} \left[\begin{array}{c} r_1,...,r_p, \frac{\alpha}{k},..., \frac{\alpha+k-1}{k}, \frac{\beta}{s},..., \frac{\beta+s-1}{s} \\ s_1,...,s_q, \frac{\alpha+\beta}{k+s},..., \frac{\alpha+\beta+k+s-1}{k+s} \end{array} ; \frac{k^k s^s \lambda t^{k+s}}{k+s} \right] \tag{4.10}$$

where λ is a constant.

Proof (i): Let left hand side of (4.9) be denoted by I and using (2.1)and (3.5), then

$$I = \int_0^t x^{\alpha-1}(t-x)^{\beta-1} \ {}_pSupersinh_q \left[\begin{array}{c} r_1,...,r_p \\ s_1,...,s_q \end{array} ; \lambda x^k (t-x)^s \right] dt$$

$$= \int_0^t x^{\alpha-1}(t-x)^{\beta-1} \frac{1}{2} \left[{}_pF_q \left[\begin{array}{c} r_1,...,r_p \\ s_1,...,s_q \end{array} ; \lambda x^k (t-x)^s \right] - {}_pF_q \left[\begin{array}{c} r_1,...,r_p \\ s_1,...,s_q \end{array} ; -\lambda x^k (t-x)^s \right] \right] dt$$

$$= t^{\alpha+\beta-1} \frac{1}{2} B(\alpha,\beta) \left[{}_{p+k}F_{q+k} \left[\begin{array}{c} r_1,...,r_p, \frac{\alpha}{k},..., \frac{\alpha+k-1}{k}, \frac{\beta}{s},..., \frac{\beta+s-1}{s} \\ s_1,...,s_q, \frac{\alpha+\beta}{k+s},..., \frac{\alpha+\beta+k+s-1}{k+s} \end{array} ; \frac{k^k s^s \lambda t^{k+s}}{k+s} \right] \right.$$

$$\left. - {}_{p+k}F_{q+k} \left[\begin{array}{c} r_1,...,r_p, \frac{\alpha}{k},..., \frac{\alpha+k-1}{k}, \frac{\beta}{s},..., \frac{\beta+s-1}{s} \\ s_1,...,s_q, \frac{\alpha+\beta}{k+s},..., \frac{\alpha+\beta+k+s-1}{k+s} \end{array} ; -\frac{k^k s^s \lambda t^{k+s}}{k+s} \right] \right]$$

$$= B(\alpha,\beta) t^{\alpha+\beta-1} \ {}_{p+k}Supersinh_{q+k} \left[\begin{array}{c} r_1,...,r_p, \frac{\alpha}{k},..., \frac{\alpha+k-1}{k}, \frac{\beta}{s},..., \frac{\beta+s-1}{s} \\ s_1,...,s_q, \frac{\alpha+\beta}{k+s},..., \frac{\alpha+\beta+k+s-1}{k+s} \end{array} ; \frac{k^k s^s \lambda t^{k+s}}{k+s} \right]$$

Similarly, we can prove (ii).

Theorem 4.4. If $Re(\alpha) > 0, Re(\beta) > 0, Re(r_1) > 0, ..., Re(r_p) > 0, Re(s_1) > 0, ..., Re(s_q) > 0$ and $Re(\sum_{k=1}^q s_k - \sum_{k=1}^p r_k) > 0$, then

$$i) \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} \cdot_p Supersinh_q \left[\begin{array}{c} r_1,...,r_p \\ s_1,...,s_q \end{array} ; x \right] dx = B(\alpha,\beta)_{p+1}Supersinh_{q+1} \left[\begin{array}{c} r_1,...,r_p,\alpha \\ s_1,...,s_q,\alpha+\beta \end{array} ; 1 \right] \tag{4.11}$$

$$ii) \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} \cdot_p Supercosh_q \left[\begin{array}{c} r_1,...,r_p \\ s_1,...,s_q \end{array} ; x \right] dx = B(\alpha,\beta)_{p+1}Supercosh_{q+1} \left[\begin{array}{c} r_1,...,r_p,\alpha \\ s_1,...,s_q,\alpha+\beta \end{array} ; 1 \right] \tag{4.12}$$

Proof:

Let left hand side of (4.11) be denoted by I, and using (2.1) and (3.6), then

$$I = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} \cdot_p Supersinh_q \left[\begin{array}{c} r_1,...,r_p \\ s_1,...,s_q \end{array} ; x \right] dx$$

$$= \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\frac{1}{2}\left[{}_pF_q\left[\begin{array}{c} r_1,...,r_p \\ s_1,...,s_q \end{array};x\right] -{}_p F_q\left[\begin{array}{c} r_1,...,r_p \\ s_1,...,s_q \end{array};-x\right]\right]dx$$

$$= B(\alpha,\beta)\frac{1}{2}\left[{}_{p+1}F_{q+1}\left[\begin{array}{c} r_1,...,r_p,\alpha \\ s_1,...,s_q,\alpha+\beta \end{array};1\right] -{}_{p+1} F_{q+1}\left[\begin{array}{c} r_1,...,r_p,\alpha \\ s_1,...,s_q,\alpha+\beta \end{array};-1\right]\right]$$

$$= B(\alpha,\beta)_{p+1}Supersinh_{q+1}\left[\begin{array}{c} r_1,...,r_p,\alpha \\ s_1,...,s_q,\alpha+\beta \end{array};1\right] = RHS$$

5 Some Special Cases

i. Putting $p=3, q=2, r_1=r_2=r_3=1$ and $s_1=s_2=2$ in (4.1) and (4.2), we have

$$a.\quad {}_3Supersinh_2\left[\begin{array}{c} 1,1,1 \\ 2,2 \end{array};x\right] = \frac{x}{2^2}+\frac{x^3}{4^2}+\frac{x^5}{6^2}+...$$

$$b.\quad {}_3Supercosh_2\left[\begin{array}{c} 1,1,1 \\ 2,2 \end{array};x\right] = 1+\frac{x^2}{3^2}+\frac{x^4}{5^2}+\frac{x^6}{7^2}+...$$

ii. Putting $\alpha=\beta=1, p=1, q=0, \lambda=1, r_1=1, k=1$ in (4.7) and (4.8), we have

$$a. \int_0^x {}_1Supersinh_0\left[\begin{array}{c} 1 \\ - \end{array};t\right]dt = \frac{x^2}{2}+\frac{x^4}{4}+\frac{x^6}{6}+...$$

$$b. \int_0^x {}_1Supercosh_0\left[\begin{array}{c} 1 \\ - \end{array};t\right]dt = x+\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}+... = tanh^{-1}(x)$$

where $-1 < x < 1$.

iii. Putting $\alpha=1, \beta=1, p=1, q=0$ and $r_1=1$in (4.11) and (4.12) then

$$a. \int_0^1 {}_1Supersinh_0\left[\begin{array}{c} 1 \\ - \end{array};t\right]dt = \frac{1}{2}+\frac{1}{4}+\frac{1}{6}+...$$

$$b. \int_0^1 {}_1Supersinh_0\left[\begin{array}{c} 1 \\ - \end{array};t\right]dt = 1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+...$$

6 Conclusion

In this study, we have presented some theorems related to the integrals of hypergeometric superhyperbolic functions, all based on generalized hypergeometric functions. We also identified specific cases where these integrals reduce to more compact and simple expressions. These findings may be highly useful and relevant to disciplines related to mathematical physics, engineering, and computational mathematics.

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