

# Linearity Property of the Difference Sequence Space $\ell_M(X, \Delta, \alpha, P, L)$ Defined by Orlicz Function M

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## Abstract

The idea of difference sequence space is introduced by Kizmaz. Lindenstrauss and Tzafriri used the concept Orlicz function M to construct the sequence space  $\ell_M$ . In this paper, we discuss the linearity property of the class  $\ell_M(X, \Delta, \alpha, P, L)$  of generalized form of difference sequence space which was introduced by Kaphle et. al.

**Keywords:** Sequence space, Orlicz function, Orlicz sequence space, Difference sequence space.

## Introduction

Sequence spaces have a major role in mathematics, especially in functional analysis and mathematical analysis. They give researchers a framework for examining infinite numerical sequences and provide information on their convergence, summability, and other characteristics. Mathematicians have extensively researched sequence spaces, which are vector spaces of sequences with elements in real or complex numbers ( $\mathbb{R}$  or  $\mathbb{C}$ ). Many mathematicians in classical analysis have investigated these spaces, which include convergence, bounded, null, and  $l_p$  spaces.

Sequence spaces emerged in early 1900's along with the development of Banach space and summability theory. Works towards Cesaro summability, Holder's sum gave a fruitful contribution to sequence space. David Hilbert (1906) introduced  $l_2$  space, which leads to the space of square – summable sequences:  $l_2 = \{\bar{x} = (x_k): \sum_{k=1}^{\infty} |x_k|^2 < \infty\}$ . Stephen Banach then introduced the absolutely p – summable sequence space  $l_p$  in 1932 as  $l_p = \{\bar{x} = (x_k): \sum_{k=1}^{\infty} |x_k|^p < \infty\}$ . This sequence space was just the generalization of  $l_2$  space which is the turning point towards the development of sequence spaces.

If  $\omega$  denotes the set of all functions from the set of positive integers  $\mathbb{N}$  to the field  $\mathbb{K}$  of real or complex numbers then it become a vector space.

Any sequence space is defined as a set of all sequences  $x = (x_k)$  which is a linear subspace of  $\omega$  over the field  $\mathbb{C}$  with the usual operations defined as

- i.  $(x_n) + (y_n) = (x_n + y_n)$
- ii.  $\lambda(x_n) = (\lambda x_n)$ .

The well-known classical sequences are null, bounded, convergent and absolutely  $p$ -summable sequence spaces which are defined by

- i.  $c_0 = \{ \bar{x} = (x_k) \in \omega : x_k \rightarrow 0 \text{ as } k \rightarrow \infty \}$  (null sequence space)
- ii.  $c = \{ \bar{x} = (x_k) \in \omega : \exists l \in X \text{ s.t. } |x_k - l| \rightarrow 0 \text{ as } k \rightarrow \infty \}$  (convergent sequence space)
- iii.  $\ell_\infty = \{ \bar{x} = (x_k) \in \omega : \sup_k |x_k| < \infty \}$  (bounded sequence space)
- iv.  $\ell_p = \{ \bar{x} = (x_k) \in \omega : \sum_{k=1}^{\infty} |x_k|^p < \infty, 0 < p < \infty \}$  (absolutely  $p$ -summable sequence space)

Maddox generalized the classical sequence spaces to the vector valued sequence spaces whose elements are the vectors from the linear space  $X$ . He defined the sequence spaces as

- i.  $c_0(X) = \{ \bar{x} = (x_k) \in X : \|x_k\| \rightarrow 0 \text{ as } k \rightarrow \infty \}$
- ii.  $c(X) = \{ \bar{x} = (x_k) \in X : \exists l \in X \text{ s.t. } \|x_k - l\| \rightarrow 0 \text{ as } k \rightarrow \infty \}$
- iii.  $\ell_\infty(X) = \{ \bar{x} = (x_k) \in X : \sup_k \|x_k\| < \infty \}$
- iv.  $\ell_p(X) = \{ \bar{x} = (x_k) \in X : \sum_{k=1}^{\infty} \|x_k\|^p < \infty, 0 < p < \infty \}$ .

Several researchers, including Kathman and Gupta [4] (1980), Maddox [7] (1981), Ruckle [14] (1981), and Malkowski and Rakocevic[8] (2004), have made substantial contributions to the theory of vector and scalar valued sequence spaces using Banach spaces.

## Preliminaries and Definitions

### a. Orlicz function:

A continuous, non-decreasing and convex function  $M: [0, \infty)$  is called orlicz function if  $M(0) = 0$ ,  $M(t) > 0$  for  $t > 0$  and  $M(t) \rightarrow \infty$  as  $t \rightarrow \infty$

For Example; The function  $M$  defined by  $M(t) = |t|^n, n > 1, t \in \mathbb{R}$  is an Orlicz function.

If convexity of Orlicz function  $M$  is replaced by  $M(t + u) \leq M(t) + M(u)$  then this function is called *modulus function*.

### b. Orlicz sequence space:

It is the sequence space defined the Orlicz function  $M$  and is defined by

$\ell_M = \{ \bar{x} = (x_k) \in \omega : \sum_{k=1}^{\infty} M(\frac{|x_k|}{\rho}) < \infty \text{ for some } \rho > 0 \}$  together with the norm defined by  $\|\bar{x}\| = \inf \{ \rho > 0 : \sum_{k=1}^{\infty} M(\frac{|x_k|}{\rho}) \leq 1 \}$ .

This space forms a Banach Space and is related to the sequence space  $\ell_p$  with  $M(t) = t^p, 1 \leq p < \infty$ . This space was first introduced by Lindenstrauss and Tzafriri [6] in 1971.

### c. Difference sequence spaces:

The concept of difference sequence spaces was first introduced by Kizmaz [7] in 1981. He defined Banach space  $X$ -valued sequence spaces

- i.  $c_0(\Delta(X)) = \{ \bar{x} = (x_k) \in X : \|\Delta x_k\| \rightarrow 0 \text{ as } k \rightarrow \infty \}$
- ii.  $c(\Delta(X)) = \{ \bar{x} = (x_k) \in X : \exists l \in X \text{ s.t. } \|\Delta x_k - l\| \rightarrow 0 \text{ as } k \rightarrow \infty \}$
- iii.  $\ell_\infty(\Delta(X)) = \{ \bar{x} = (x_k) \in X : \sup_k \|\Delta x_k\| < \infty \}$
- iv.  $\ell_p(\Delta(X)) = \{ \bar{x} = (x_k) \in X : \sum_{k=1}^\infty \|\Delta x_k\|^p < \infty, 0 < p < \infty \}$

where  $\Delta x_k = x_k - x_{k-1}$ .

These spaces are Banach spaces with norm  $\|x\|_\Delta = |x_1| + \|\Delta x\|_\infty$ .

Et and Kolak[4], in 1995, generalized the concept of Kizmaz to study the  $\Delta^m$  - sequence spaces of Banach space  $X$ -valued sequences and later they defined the following classes

- i.  $c_0(\Delta^m(X)) = \{ \bar{x} = (x_k) \in X : \|\Delta^m x_k\| \rightarrow 0 \text{ as } k \rightarrow \infty \}$
- ii.  $c(\Delta^m(X)) = \{ \bar{x} = (x_k) \in X : \exists l \in X \text{ s.t. } \|\Delta^m x_k - l\| \rightarrow 0 \text{ as } k \rightarrow \infty \}$
- iii.  $\ell_\infty(\Delta^m(X)) = \{ \bar{x} = (x_k) \in X : \sup_k \|\Delta^m x_k\| < \infty \}$

where  $m \in \mathbb{N}, \Delta^0 x = (x_k), \Delta x_k = x_k - x_{k-1}$ , and  $\Delta^m x_k = \Delta^{m-1} x_k - \Delta^{m-1} x_{k-1}$

and so that  $\Delta^m x_k = \sum_{r=1}^m (-1)^r \binom{m}{r} x_{k+r}$  and showed that these classes are Banach spaces with norm defined as  $\|x\|_\Delta = \sum_{r=1}^m |x_r| + \|\Delta^m x\|_\infty$ .

Also, in 2006, Tripathy and Esi [17] studied a new type of difference sequence spaces  $c(\Delta_m), c_0(\Delta_m), c(\Delta_m)$  where  $m \in \mathbb{N}$  as defined by

$$Z(\Delta_m) = \{ \bar{x} = (x_k) \in \omega : \Delta_m x \in Z \}, \text{ for } Z = \ell_\infty, c, c_0$$

where,  $\Delta_m x = (\Delta_m x_k) = (x_{k+m} - x_k)$  for all  $k \in \mathbb{N}$ .

For  $m=1$ ,  $\ell_\infty(\Delta_m) = \ell_\infty(\Delta)$ ,  $c(\Delta_m) = c(\Delta)$ ,  $c_0(\Delta_m) = c_0(\Delta)$ .

They proved that these spaces are Banach space with norm defined by

$$\|x\|_\Delta = \sum_{r=1}^m |x_r| + \sup_k |\Delta_m x_k|.$$

In 2012, Srivastava and Pahari [15] introduced the following class  $\ell_M(X, \alpha, P) = \{ x = (x_k) \in \omega(X) : \sum_{k=1}^\infty M \left( \frac{\|\alpha_k x_k\|^{p_k}}{\rho} \right) < \infty \text{ for some } \rho > 0 \}$  and  $\ell_M(X, \alpha, P, L) = \left\{ x = (x_k) \in \omega(X) : \sum_{k=1}^\infty M \left( \frac{\|\alpha_k x_k\|^{p_k/L}}{\rho} \right) < \infty \text{ for some } \rho > 0 \right\}$ , where  $\sup_k p_k = L < \infty$ .

For  $a = (a_k)$  for a sequence of non zero complex numbers and  $P = (p_k)$  for any sequenced of strictly positive real numbers. They also investigated conditions for containment relation  $\ell_M(X, \alpha, P)$  and explore the linear topological structure of the class  $\ell_M(X, \ell, \alpha, P, L)$

Many researchers, including Subramanian (2008)[18], Faried and Bakery (2009)[11], Basar and Yesilkayagil(2019)[2] etc. have used Orlicz sequence space to construct the different difference sequence spaces. They investigated the structural features and inclusion relationships between the spaces. Paudel and et al [12][13][14], Pokharel [15]studies the various properties of sequence space using fuzzy concept.

For  $a = (a_k)$  for a sequence of non-zero complex number and  $P = (p_k)$  for any sequence of strictly positive real numbers, and  $L \sup_k p_k < \infty$  Kaphle and et al [5] defined the classes of sequences  $\ell_M(X, \Delta, \alpha, P)$  and  $\ell_M(X, \Delta, \alpha, P, L)$  as the generalization of the sequence spaces  $\ell_M(X, \alpha, P)$  and  $\ell_M(X, \Delta, \alpha, P)$  defined by and discussed by Srivastava and Pahari [15] to the difference sequence space as:

$$\ell_M(X, \Delta, \alpha, P) = \left\{ x = (x_k) \in \omega(X) : \sum_{k=1}^{\infty} M\left(\frac{\|\alpha_k \Delta_m x_k\|^{p_k}}{\eta}\right) < \infty \text{ for some } \eta > 0 \right\}$$

and

$$\ell_M(X, \Delta, \alpha, P, L) = \left\{ x = (x_k) \in \omega(X) : \sum_{k=1}^{\infty} M\left(\frac{\|\alpha_k \Delta_m x_k\|^{p_k/L}}{\eta}\right) < \infty \text{ for some } \eta > 0 \right\},$$

where  $\Delta_m x_k = x_{k+m} - x_k$ ,  $m$  is a positive integer.

Clearly,  $\ell_M(X, \Delta, \alpha, P, L)$  is a subset of  $\ell_M(X, \Delta, \alpha, P)$ .

Throughout the article, we shall denote for  $\alpha = (\alpha_k)$ ,  $\beta = (\beta_k)$

$$\gamma_k = \frac{\alpha_k}{\beta_k} \text{ and } \delta_k = \left| \frac{\alpha_k}{\beta_k} \right|^{p_k}.$$

## Main Results

Based on the class of sequence we study about the linearity properties of these spaces.

**Lemma 3.1** [1]: Let  $(p_k)$  be a bounded sequence of strictly positive real numbers with  $0 < p_k \leq \sup p_k = L$ ,  $D = \max\{1, 2^{-1}\}$  then

- i.  $|x + y|^{p_k} \leq D\{|x|^{p_k} + |y|^{p_k}\}$ ;
- ii.  $|\alpha|^{p_k} \leq \max(1, [\alpha]^L)$ .

**Lemma 3.2** [5].  $\ell_M(X, \alpha, P) \subset \ell_M(X, \beta, P)$  if and only if  $\inf_k \delta_k > 0$ .

**Lemma 3.3** [5].  $\ell_M(X, \beta, P) \subset \ell_M(X, \alpha, P)$  if and only if  $\limsup_k \delta_k < \infty$ .

**Theorem 3.4.** The space  $\ell_M(X, \Delta, \alpha, P)$  is linear if and only if  $\sup_k \delta_k < \infty$ .

Proof: Firstly assume  $\sup_k \delta_k < \infty$ . Let  $x, y \in \ell_M(X, \Delta, \alpha, P)$ .

Then there exists  $\eta_1, \eta_2$  such that

$$\sum_{k=1}^{\infty} M\left(\frac{\|\alpha_k \Delta_m x_k\|^{p_k}}{\eta_1}\right) < \infty$$

and

$$\sum_{k=1}^{\infty} M\left(\frac{\|\alpha_k \Delta_m y_k\|^{p_k}}{\eta_2}\right) < \infty.$$

Take any two complex numbers  $z$  and  $w$ . We show that  $zx + wy \in \ell_M(X, \Delta, \alpha, P)$ . Choose,  $\eta_3 > 0$  such that

$$2D \eta_1 \max(1, |z|) \leq \eta_3 \quad \text{and} \quad 2D \eta_2 \max(1, |w|) \leq \eta_3.$$

Since  $M$  is non-decreasing and convex,

$$\begin{aligned} \sum_{k=1}^{\infty} M\left(\frac{\|\alpha_k (z \Delta_m x_k + w \Delta_m y_k)\|^{p_k}}{\eta_3}\right) &\leq \sum_{k=1}^{\infty} M\left(\frac{D\|z\alpha_k \Delta_m x_k\|^{p_k} + D\|w\alpha_k \Delta_m y_k\|^{p_k}}{\eta_3}\right) \\ &= \sum_{k=1}^{\infty} M\left(\frac{D|z|^{p_k}\|\alpha_k \Delta_m x_k\|^{p_k} + D|w|^{p_k}\|\alpha_k \Delta_m y_k\|^{p_k}}{\eta_3}\right) \leq \sum_{k=1}^{\infty} M\left(\frac{\|\alpha_k \Delta_m x_k\|^{p_k}}{2\eta_1} + \frac{\|\alpha_k \Delta_m y_k\|^{p_k}}{2\eta_2}\right) \\ &\leq \frac{1}{2} \sum_{k=1}^{\infty} M\left(\frac{\|\alpha_k \Delta_m x_k\|^{p_k}}{\eta_1}\right) + \frac{1}{2} \sum_{k=1}^{\infty} M\left(\frac{\|\alpha_k \Delta_m y_k\|^{p_k}}{2\eta_2}\right) < \infty. \end{aligned}$$

Hence,  $zx + wy \in \ell_M(X, \Delta, \alpha, P)$ . That is  $\ell_M(X, \Delta, \alpha, P)$  is linear.

**Conversely suppose that**

$\ell_M(X, \Delta, \alpha, P)$  is linear but  $\sup \delta_k = \infty$ . Then there exists sequence  $(k(n))$  of integers such that  $k(n+1) > k(n) \geq 1$  for which  $\left|\frac{\alpha_{k(n)}}{\beta_{k(n)}}\right|^{p_{k(n)}} > n^2, \quad \forall n \geq 1.$

Now corresponding to  $z \in X$ , with  $\|z\| = 1$  we can define a sequence  $x = (x_k)$  by

$$\Delta_m x_k = \begin{cases} \beta_{k(n)}^{-1} n^{-2/p_{k(n)}} z & \text{for } k = k(n), \quad n \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

For  $k = k_n, n \in \mathbb{N}$ , since  $M$  is convex,

$$\begin{aligned} \sum_{k=1}^{\infty} M\left(\frac{\|\alpha_k \Delta_m x_k\|^{p_k}}{\eta}\right) &= \sum_{k=1}^{\infty} M\left(\frac{\|n^{-2/p_{k(n)}} z\|^{p_{k(n)}}}{\eta}\right) \\ &= \sum_{k=1}^{\infty} M\left(\frac{1}{\eta n^2}\right) \\ &\leq M\left(\frac{1}{\eta}\right) \sum_{k=1}^{\infty} \frac{1}{n^2} < \infty. \end{aligned}$$

So that  $x \in \ell_M(X, \Delta, \alpha, P)$ .

On the other hand, for any  $\eta > 0$ , taking a scalar  $a = 4$ , we have

$$\sum_{k=1}^{\infty} M\left(\frac{\|\alpha_k a \Delta_m x_k\|^{p_k}}{\eta}\right) = \sum_{k=1}^{\infty} M\left(\frac{\|4 n^{-2/p_{k(n)}} z\|^{p_{k(n)}}}{\eta}\right)$$

$$\geq \sum_{k=1}^{\infty} M\left(\frac{4^n}{\eta n^2}\right).$$

since  $\frac{4^n}{n^2} > 1$  for each  $n \geq 1$ , we have

$$\sum_{k=1}^{\infty} M\left(\frac{4^n}{\eta n^2}\right) \geq \sum_{k=1}^{\infty} M\left(\frac{1}{\eta}\right) = \infty,$$

which shows that

$$ax \notin \ell_M(X, \Delta, \alpha, P),$$

which gives a contradiction. Hence  $\sup_k \delta_k < \infty$ .

**Theorem 3.5**  $\ell_M(X, \Delta, \alpha, P, L)$  forms a linear space over  $\mathbb{C}$ .

Proof: Since  $\ell_M(X, \Delta, \alpha, P, L)$  is a subset of the space  $\ell_M(X, \Delta, \alpha, P)$  and since we have assumed  $L = \sup_k p_k < \infty$  is finite. Then from Theorem 3, we can see that  $\ell_M\left(\begin{matrix} X, \Delta, \\ \alpha, P, L \end{matrix}\right)$  forms a linear space over  $\mathbb{C}$

## Conclusion

From the above mentioned theorem, we can conclude that  $\ell_M(X, \Delta, \alpha, P, L)$  is a linear difference sequence space defined by the Orlicz function  $M$ .

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