Learning Difficulties with Finite Geometries and Remedial Teaching

Uday Raj Joshi

Abstract

Students face many difficulties in learning and studying finite geometries. Geometry is considered as a hard subject from lower levels. In lower levels, even upto higher secondary level, students are taught mainly the Euclidian geometry. In B. Ed level, it is necessary to introduce them with different types of geometries such as finite geometries, Euclidian, non-Euclidian, projective geometry, topology etc. The aim of this research is to find the reasons and facts which make finite geometries hard to understand. There are many types of problems that create these difficulties. One of the important problems is that most of the students have negative attitude towards geometry that it is a hard subject. Most of them confuse finite geometries with the Euclidian geometry. Some of them have wrong concept that Euclidian geometry is only the geometry that represents the real universe. The problem is on the foundation of the geometry. The little knowledge and understanding about what is geometry? What is an axiomatic system? make serious learning difficulties in finite geometries.

Key words: Learning difficulties, finite geometries, axiomatic system, concrete models, and remedial teaching.

Introduction

Students read some sort of Euclidian geometry and analytic geometry in lower classes, high school level and higher secondary level. In B. Ed. level of Trivuwan University, they are introduced with different types of geometries like finite geometries, projective geometry, non-Euclidian geometry, neutral geometry and topology. Most of the students feel difficult to understand these geometries than the analytical geometry containing two dimensional and three dimensional geometries. Out of this vast topic, only finite geometry is taken. The aim of this research is to find the reasons, difficulties, problems and facts which make finite geometries hard to understand.

A geometry which contains only a finite number of points, lines and planes is known as a finite geometry. Such geometry, in general, has a small number of axioms and theorems. Finite geometries provide us an opportunity to study geometries of relatively simple structures using the axiomatic method. Gino Fano, who worked mainly in the areas of projective and algebraic geometry, was the first person to consider the notion of a finite geometry, one that was a three dimensional and contained 15 planes, 35 lines and 15 points, each plane containing 7 points and 7 lines. Four-point geometry, four-line geometry, Fano's geometry, three-point geometry and Young's geometry are some important examples of finite geometries.

Euclid wrote "The Elements", by collecting and compiling the mathematics developed up to the time. It was about 2200 years ago. He started by stating his assumptions. By stating his assumptions, he gave rigor to his arguments. By focusing on the logical reasoning that goes into problem solving, Euclid put the method of solving a problem, and not merely the solution, into the spotlight. Euclid had five common notions and five axioms. Actually, in Euclid's time the word axiom was reserved for something obvious, a common notion, while postulate meant something to be assumed. However, in present day language we use the word axiom to mean something that is assumed. Hence, we will always use the modern terminology.

An axiom is known as a statement that is accepted without proof. Euclid's five axioms can be written as follows:

1. A line can be drawn from a point to any other point.
2. A finite line can be extended indefinitely.
3. A circle can be drawn, given a center and a radius.
4. All right angles are equal.
5. If a line intersects two other lines such that the sum of the interior angles on one side of the intersecting line are less than the sum of two right angles, then the lines meet on that side and not on the other side.

The sort of geometry that Euclid wrote about takes place on plane. While 'The Elements' may be the most successful textbook ever written, with over one-thousand editions and over two-thousand years of usage, there is still room for improvement. In the early 20th century, mathematicians pointed out that there are some logical flaws in the proofs which Euclid gives. David Hilbert, one of the great mathematicians of the 20th century, required around 20 axioms to prove all the theorems in The Elements. Nevertheless most of the theorems in The Elements are proved more-or-less correctly, and the text continues to have influence to this day. And the geometry thus developed is known as Euclidean Geometry in honor of Euclid.

The first four axioms are easy to understand, but the fifth is more complex and lengthy than any other axioms. Many mathematicians thought that this could be proved on the basis of remaining axioms. On this process other geometries like hyperbolic, elliptic were evolved.

There are some important differences between finite geometries and Euclidian geometry. A finite geometry contains only a finite number of points, lines and planes while the Euclidian geometry contains infinite number of points, and lines. Finite geometry, in general, has a small number of axioms and theorems but in Euclidian geometry, there are many axioms, many more definitions and highly large number of theorems. Finite geometries provide us an opportunity to study geometries of relatively simple structures while Euclidian geometry is relatively complex.

Methodology

The research was taken place on the students of mathematics reading at B. Ed. second year at Surkhet Campus (Education) on March, 2013. Geometry is a compulsory subject in Second Year for the students taking mathematics as a major subject. Geometry in this level is divided mainly in two parts: analytical geometry and roads to geometry. The second part introduces different types of geometry. The aim of the research was to find the learning difficulties faced by those students while studying finite geometries and to purpose a way of remedial teaching. Both qualitative and quantitative methods were used for the purpose. Interviews were taken before and after the classes. Self experience and discussion with the respective teachers was used. From the class of about 60, 20 boy and 10 girl (in proportion) students were chosen by lottery method to fill up the questionnaire.

Analysis and Interpretation

To interpret and analyze the data and results obtained from the research, it is relevant to present a finite geometry and its difficulties and possible way of remedial teaching with reference to it. So some fundamental concepts and Four-point geometry are given below:

Axiomatic System

Axiomatic method is used in the development of all of modern mathematics like algebra, mathematical analysis, topology, geometry etc. and such a system contains some undefined terms or primitive terms, axioms or postulates, some defined terms and theorems or facts.

If we try to define a term, then its definition contains one or more new terms which in turn must be defined. This process forms either a circular chain or a linear unending chain as shown in the following figure:

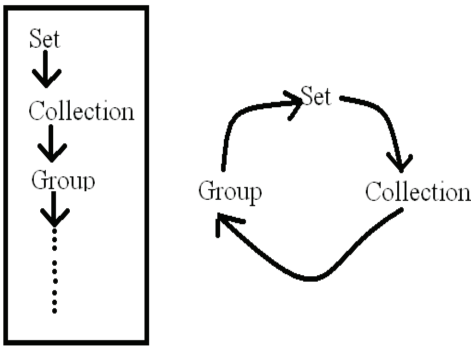

Figure1: Axiomatic System

The unending infinite chain is not acceptable because of the obvious reason. The circulatory is also unacceptable because it at last clarifies nothing (set means a collection, collection means group, group means set and hence set means a set ending at the same thing). So the collection of definitions must end at some point, and one or more of the terms will remain undefined. These terms are known as the undefined or primitive terms of the axiomatic system. Other terms in the system are defined in terms of the primitive terms and are called defined terms.

The primitive terms and definitions can now be combined into the statements or facts or theorems of the axiomatic system. For these theorems or facts, we must supply logically deduced proofs of their validity. We now need additional statements to prove these theorems, which in tern require proof. As above, we form a chain of statements, either circulatory or infinitely large. To avoid this impractical structure, one or more of these statements must remain unproved and accepted to be true by our intuition. These statements are called axioms or postulates of the axiomatic system.

It is not necessary that undefined terms, defined terms, axioms or theorems have some meaning in the real life.

Models

As discussed above, each axiomatic system contains a number of undefined terms. Since these terms are truly undefined, they have no inherent meaning, and each may choose one or more way to interpret them. By giving each undefined term in a system a particular meaning, we create an interpretation of the system. If, for a given interpretation of a system, all the axioms are correct, we call the interpretation as a model.

Models produce validity to the systems and make easy to understand the system. If someone tells 1+1=2, 1+0=1, 0+1=1 and 0+0=0, then there is no necessity to explain but what if one say 1+1=1, 1+0=0, 0+1=0, 0+0=0, where 1and 0 are undefined terms. For many people with little knowledge of mathematics it seems wrong, unbelievable and absurd. But they surprise and believe on the system if we present the model of the system as below:

In this model, A and B are switches that can be on or off to pass current, C is the battery as a source of electricity and D is the bulb. First number represents first switch A and second number represents switch B. 1 represents for switch on and 0 represents for switch off . The result is 1 if the bulb produces light. Otherwise the result is 0. For example 1+0=0 means that when the first switch A is on and B is off, the bulb produces no light. Similarly, the following algebra can be described:

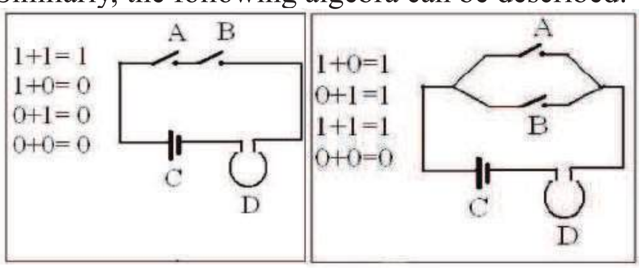

Figure2: Model

Properties of Axiomatic Systems

The most important and most fundamental property of an axiomatic system is consistency. A system of axioms is said to be consistent if it is impossible to deduce from these axioms a theorem that contradicts any axiom or previously proved theorem. If the system is not consistent then it neither has mathematical meaning nor any importance to study. To check whether a system is consistent or not, we will make use of models.

There are two types of models: (1) concrete models, where interpretations of the undefined terms are objects or relations taken from real world, and (2) abstract models, where interpretations of the undefined terms are taken from some other axiomatic systems such as the real number system.

If the system is not consistent i.e. inconsistent, then the contradictory theorems deduced from the axioms would have contradictory counterparts in the real world, which we accept as impossible and so no concrete model for a such system exists. This proves that if a concrete model for a system exists, then the system is consistent. Existence of the concrete model for an axiomatic system is known as the absolute consistency.

However, the construction of a concrete model of an axiomatic system is not always possible. Let us consider the case in which there are infinitely many distinct undefined terms. Since the real objects in the known universe are finite, so the interpretations of the all the undefined terms cannot be the objects and hence impossible to create a concrete model. In such cases we establish a model using concepts from some other axiomatic system, whose consistency has been already established such as the system of the real numbers. Consistency formed in this way is known as relative consistency and the two axiomatic systems are said to be relatively consistent.

As discussed above it has no meaning to study more an axiomatic system without consistency. Now we discuss two more properties of an axiomatic system which are different from consistency. This difference lies in the fact that, unlike the consistency property, we don't require that axiomatic systems possess these properties to be useful (worthy of study).

An axiom is said to be independent if it cannot be logically deduced from the axioms in the system. The set of all axioms is said to be independent if each of its axioms are independent.

We say that an axiom set is of sufficient size or complete if it is impossible to add an additional consistent and independent axiom without adding additional undefined terms. If all models of an axiomatic system are isomorphic, then the set of axioms is said to be categorical. This property implies completeness.

An Abstract Example of an Axiomatic System

Undefined Terms: Fe's, Fo's, and the relation" belong to".

Axiom 1. There exist exactly three distinct Fe's in this system.

Axiom 2. Two distinct Fe's belong to exactly one Fo.

Axiom 3. Not all Fe's belong to the same Fo.

Axiom 4. Any two distinct Fo's contain at least one Fe that belongs to both.

Fe-Fo Theorem 1. Two distinct Fo's contain exactly one Fe.

Proof. Since Axiom 4 states that two distinct Fo's contain at least one Fe, we need only show that these two Fo's contain no more than one Fe. For this purpose we will use an indirect proof and assume that two Fo's share more than one Fe. The simplest case of more than one is two. Now each of these two Fe's belonging to two distinct Fo's, but that in turn contradicts Axiom 2, and we are done.

Fe-Fo Theorem 2. There are exactly three Fo's.

Proof. Axiom 2 tells us that each pair of Fe's is on exactly one Fo. Axiom 1 provides us with exactly three Fe's. Axiom 3 guarantees that the three Fe's are not on the same Fo; therefore, by counting, distinct pairs of Fe's, we find that we have at least three Fo's. Now suppose that there exist a distinct fourth Fo. Theorem 1 tells us that the fourth Fo must share a Fe with each of the other Fo's. Therefore, it must contain at least one of the two of the existing three Fo's, but Axiom 2 prohibits this. Therefore, a forth Fo cannot exists, and there are exactly three Fo's.

Fe-Fo Theorem 3. Each Fo has exactly two Fe's that belong to it.

Proof. By theorem 2, we have exactly three Fo's. Now Axiom 4 provides that each Fo has at least one Fe, and Axiom 1 prevents it from containing exactly one. Axiom 1 and Axiom 3 prevent a Fo from containing more than two Fe's.

Examples:

Concrete Model 1:

Let us designate the Fe's as people and Fo's as committees, and the axioms become the following:

Axiom 1. There are exactly three people.

Axiom 2. Two distinct people belong to exactly one committee.

Axiom 3. Not all people belong to the same committee.

Axiom 4. Any two distinct committees contain one person who belongs to both.

Let the people be Ram, Shyam and Hari, and the committees be Entertainment (Ram and shyam), Finance (Shyam and Hari), and Refreshment (Ram and Hari) as shown in the figure:

Four-Point Geometry

The four-point geometry, which, as we will see, derives its name from its first axiom, has an its undefined terms point, line, and on. The following set of three axioms will be assumed:

Axiom 1. There exists exactly four points.

Axiom 2. Any two distinct points have exactly one line on both of them.

Axiom 3. Each line is on exactly two points.

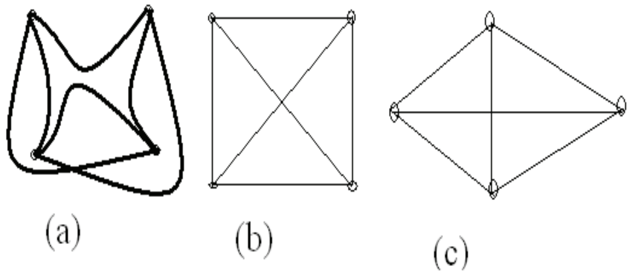

Figure3: Four-Point Geometry

Definition1 (Intersecting Lines). Two lines on the same point are said to intersect and are called intersecting lines.

Definition 2 (Parallel Lines). Two lines that do not intersect are called parallel lines.

Four-point Theorem 1. In the four point geometry, if two distinct lines intersect, then they have exactly one point in common.

Proof. By Definition 1, two distinct intersecting lines have at least one point in common, and Axiom 2 prohibits them from having more than one point in common. This completes the proof of the theorem.

Four-point Theorem 2. The four-point geometry has exactly six lines.

Proof. From Axiom 2, each pair of points has exactly one line on both of them and Axiom 1 provides four points. This means there are 4 points in total and we are taking two of them to form a line. Hence by the theory of combination, there are C(4,2) = 6 lines. Axiom 3 guarantees no more.

Four-point Theorem 3. Each point of the four-point geometry has exactly three lines on it.

Proof. By axiom 2, each point has a line in common with each of other tree points. Therefore, we have at least three lines on each point. Suppose that a fourth line was on one of the given points; then, by axiom 3, it must be one of other points but this would violate Axiom 2. Therefore, there are exactly three lines on each point.

Four-point Theorem 4. In the four point geometry, each distinct line has exactly one line parallel to it.

Proof. Axioms 1 and 3 provide us with a line l and a point P not on line l. Four-point Theorem 3 tells us that there are exactly three lines on P, and axiom 2 tells us that two of them must intersect l. Therefore, we have at least one line parallel to l. Suppose that there was a second line parallel to l. This line could not contain P without violating Four-point Theorem 3, and since it is parallel to l, it cannot contain either of the points on l. Now, either the second parallel contains only one point, which violates Axiom 3, or there exists a fifth point, which violates Axiom 1. Therefore, the second parallel line cannot exist and there exist exactly one.

Alternative Proof: Since this geometry is finite, it is possible to examine every possible case of points and lines. By using figure 1, where the points are represented by the letters A, B, C, and D and the lines by columns of letters, we may check directly to see that two distinct lines intersect in exactly one point, that there must be exactly six lines, that each point has exactly three lines on it, and each line has exactly one line parallel to it.

L1	L2	L3	L4	L5	L6
A	A	A	B	B	C
B	C	D	C	D	D

Figure4: Alternative Profe

(Wallace & West,1998]

Concrete Model of Four-Point Geometry:

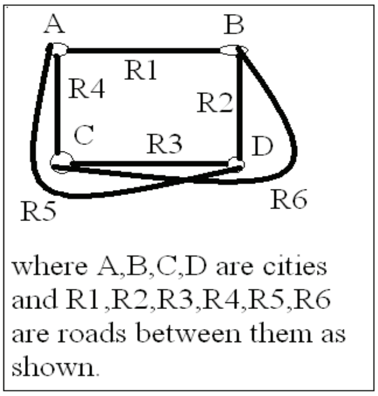

Figure5: Concrete Model of Four-Point Geometry

Main Results of the Study

The difficulties and obstacles that make finite geometries hard to understand are found as below:

1. Most of the students (86.67%) couldn't answer the basic questions such as what is geometry, what is an axiomatic system etc. They were found to be led by wrong basic concepts about geometry. It made hard to them to understand any type of geometry as an axiomatic system.
2. They confused finite geometries with the Euclidian geometry which is another serious learning difficulty. For Example: 93.33 percent students involved in the research were found to think that the concept of point and line was same in four-point geometry as in the Euclidian geometry or calculus or two or three dimensional geometry. So they were failed to understand the basic structure of four-point geometry.
3. Abstract model of finite geometries was found as another difficulty.
4. Most of the students involved in the research hoped more practical examples and concrete models of finite geometries from their teacher instead of abstract one.
5. Majority of the students (76.67%) hoped that at least one finite geometry should be taught by using solid teaching material as shown in figure7.
6. Most of the students (93.33%) hoped discussion method from their teacher instead of lecture method.
7. Why to study geometry? What should be the application of finite geometries? Since teachers didn't answer such questions in the class, so students could not be motivated to understand.
8. About 40% students found English language as one the learning difficulties. They hoped that their understanding should be better if their text are in Nepali.

Remedial Teaching

Some hints for remedial teaching are given below:

1. Teacher should provide deep knowledge of axiomatic system and geometry as an axiomatic system. He/she should use more examples from daily life and should present both concrete and abstract models as shown 6 above in four-point geometry. If students understand one of the finite geometries properly then they can understand others themselves.

2. Solid teaching material should be used in teaching first finite geometry.

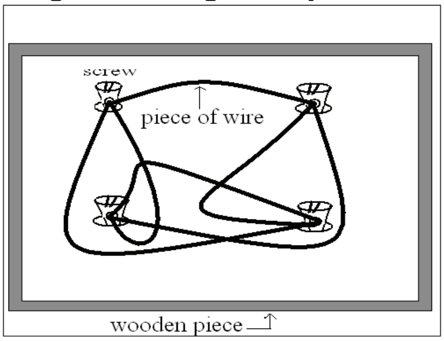

Figure6: Method of Preparing Teaching Material for 4-point geometry

3. Difference between different types of geometries such as finite geometries, Euclidian, non-Euclidian etc should be clearly stated so that no confusion arises between finite geometries and Euclidian geometry. Meaning of undefined term as point, line in finite geometry and Euclidian geometry should be stated properly. We may use examples, models and teaching material for the purpose.
4. Discussion methods or answer-question method or presentation method should be used instead of lecture method in teaching finite geometry.
5. Discussion on the applications of finite geometries should be helpful to motivate the class to study the finite geometries.
6. Examples from daily life, concrete models and solid teaching material as in figure 7 should help to understand the abstract matter of finite geometries.

Conclusions

The difficulties and obstacles faced by students in learning finite geometries are listed above under the heading "main results". Lack of basic knowledge about the axiomatic system is the main difficulty because of which students can not understand the nature, structure and philosophy of geometry. Use of teaching material as shown in figure 7 is helpful in teaching learning process. Teachers should provide their class in finite geometries in such way that they would not confuse the matter with Euclidian geometry. Teacher should understand and aware of these difficulties and should provide remedial teaching to remove these difficulties.

About the author

Mr. Joshi is working at Surkhet Campus (Education) as a Teaching Assistance since 2066/10/25. Currently he is head of Department of Mathematics and Economics Education. He is interested in the study of mathematical physics.

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