THE NEW INTEGRAL TANSFORM

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ABSTRACT

In this paper, a new integral transform was applied to solve linear ordinary differential equations with constant co-efficient. Overall, it is a fundamental tool in engineering and applied mathematics, providing a systematic way to analyze and solve dynamic systems and differential equations in a wide range of applications. Integral transforms are an effective tool for evaluating functions, resolving differential equations, and resolving issues in a variety of scientific and technical fields. Integral transform enables us to operate in other domains (such as time, frequency), where the problem's physical interpretation may be clearer or the mathematical procedures may be simpler. It is a powerful mathematical instrument that solves ordinary differential equations (ODEs) and analyzes linear time-invariant systems, among other things. It is used to investigate several problems in engineering, physics, and applied mathematics. Both ordinary differential equations and a system of linear differential equations with constant coefficients have been solved using this novel transform. Solving linear ordinary differential equations with constant coefficients is the primary use of this method. By taking the Laplace transform of both sides of an ODE, one can convert the differential equation into an algebraic equation, which is often easier to solve.

Keywords: Differential Equations, Laplace transform, Fourier transform, Elzaki transform, and Simudu transforms

1. INTRODUCTION

A function is converted from its original function space into a new function space by integration in an integral transform, where some of the original function's properties may be easier to characterize and control than in the original function space. The inverse transform can frequently be used to map the transformed function back to the original function space.

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This technique is widely used in various fields such as engineering, physics, and applied mathematics to simplify the analysis and solution of differential equations, convolution problems, and boundary value problems. The most commonly used integral transforms include the Fourier Transform, Laplace Transform, and Mellin Transform, among others. This literature review provides an overview of the key developments, applications, and theoretical advancements in the field of integral transforms.

1.1. BACKGROUND OF THE STUDY

A transformation is a tool used in mathematics that may be used to change a function (t) into a new function involving variables f(p), f(s) and so on. For about 200 years, Integral transforms have been effectively employed in the fields of mathematics and engineering. Integral transforms, such as the Fourier and Laplace transforms, have their roots in the seminal treatises of Joseph Fourier (1768–1830) and P.S. Laplace (1749–1827), who both contributed to the field of probability theory in the 1780s (Wider, 1946).

A mathematical procedure known as an integral transform transforms a given function into a new function by integrating it with a kernel function, also known as a transform kernel (Watugala, 1998).

The general form of an integral transform can be expressed as:

$$F(u) = \int_a^b K(u, x) . f(x) dx \quad \text{(i)}$$

where

F(u) is the transformed function.

K (u, x) is the transform kernel or kernel function.

f(x) is the original function.

u is the transformed variable.

x is the original variable.

The integration is performed over a suitable interval [a, b] (Spiegel,1965).

Integral transforms often have inverse transforms, which allow you to retrieve the original function from the transformed function. The inverse transform is generally given by:

$$f(x) = \int_c^d K^{-1}(x, u) . F(u) du \quad \text{(ii)}$$

where K^{-1} *(x,u)* is the inverse kernel and the integration is performed over a suitable interval [c, d]. Some well-known integral transforms include (Belgacem, Karaballi, & Kalla,2003).

a. Fourier Transform
   Fourier Transform changes a function from the time domain into frequency domain. It is used to analyze periodic and non-periodic functions.
b. Laplace Transform
   Laplace transform is used to analyze linear time-invariant systems and also solve ordinary differential equations.
c. Mellin Transform
   Mellin transform is used to generalize the Laplace transform, often used in complex analysis and number theory.
d. Hankel Transform
   Hankel transform is used in cylindrical coordinate systems and in problems involving cylindrical symmetry.
e. Radon Transform
   Radon transform is used in medical imaging, particularly in the field of computed tomography (CT) scans.
f. Integral Transforms in Probability
   The moment-generating function and characteristic function are examples of integral transforms used in probability and statistics (Kumar, 2022, 2023, &2024).

2. PROBLEM STATEMENT

Ordinary differential equations (ODEs) are fundamental in describing various physical, engineering, and biological processes. Traditional methods for solving ODEs, such as separation of variables or integrating factors, can be cumbersome for complex equations. The new transform offers a robust alternative, particularly for linear ODEs with constant coefficients and initial value problems. This transform converts differential equations into algebraic equations, making them easier to solve.

3. OBJECTIVE

To solve the ordinary differential equation by new transform with examples.

4. SIGNIFICANCE OF STUDY

Integral transforms remain a vital area of research and application in mathematics and engineering. Their ability to simplify complex problems and provide insightful solutions continues to drive advancements in various scientific and technological fields. Ongoing research is likely to uncover new applications and theoretical insights, further solidifying the importance of integral transforms in modern science and technology.

Integral transforms, including Fourier, Laplace, and Mellin, are foundational tools in applied mathematics and engineering. Their theoretical elegance and practical utility continue to drive advancements across a wide array of disciplines. Recent developments in computational methods and applications in emerging technologies ensure that integral transforms remain at the forefront of scientific research and innovation.

4.1. JUSTIFICATIONS

The solution of ordinary differential equations using the recently created "NEW TRANSFORM" is demonstrated in this work. This recently created technique is really simple to use and requires little time to implement with less efforts.

5. LITERATURE REVIEW

A function specified on the real numbers is transformed into a complex function defined on the complex plane using the Laplace transform, which is an integral transform. It is an effective mathematical tool used to examine many issues in engineering, physics, and applied mathematics as well as it analyzes linear time-invariant systems and solve ordinary differential equations (ODEs).

The Laplace transform is another individual bearing the name Pierre-Simon, marquis de Laplace (1749–1827). It transforms a lot of fascinating characteristics. Over the set of functions, this transform is defined (Elzaki, Elzaki & Hilal , 2012).

$$A = \{f(t): \exists\ M, \tau 1, \tau 2 > 0, |f(t)| < Me\ t\ \tau, \text{ if } t \in (-1)\ j \times [0, \infty)\}) \qquad \text{(iii)}$$

The Laplace transform of time function F(t) denoted by S [F(t) : u]

$$L[f(t)] = F(s) = \int_0^\infty e^{-st} f(t) d(t),\ (0 < t < \infty) \qquad \text{(iv)}$$

s is a parameter that can be either real or complex and is unrelated to t.

The New Transform and its core features are described based on the mathematical simplicity of the concept. The author of this article introduced this transform to make solving ordinary and partial differential equations simple. With appropriate examples, the basic characteristics of this transform are discussed (Widder, 1946).

Integral transforms provide a powerful tool for solving differential equations, analyzing functions, and solving problems in various scientific and engineering disciplines. They allow you to work in different domains (e.g., time, frequency) where the mathematical operations might be simpler or where the physical interpretation of the problem is clearer (Koklu, 2021).

Since non-linear differential equations predominate, it is difficult to arrive at an accurate or analytical solution. A suitable transform is required to handle non-linear terms without any difficulties and with small computation sizes, the Laplace transform, from which this transform is derived (Vallee, & Soones, 2004).

6. METHODOLOGY

This transform is very much close to Laplace transform, so it useful in solving like Laplace transform. It is also applicable where v =1 for unit variable.

Apply this new transform to both sides of the ordinary differential equation, solve the resulting algebraic equation for F(s). Then apply the inverse transform to find the solution f(t).

7. MAIN DISCUSSION AND RESULTS

The function f(t) in the set A is defined by this new transform, which is also defined for functions of exponential arrays.

$$A=\{f(t):\ \exists M,\ k_1 and k_{2>0}: |f(t)| < Me^{|t|}\ j,\qquad t \in -1 \times (0,\infty)\} \qquad (v)$$

The New transform denoted as A which is defined by the integral equations.

$$N[f(t)] = N = \int_0^\infty f(t)\, e^{-t} dt \qquad t > 0 \qquad (vi)$$

The Laplace transform and this transform are quite close. This transform will be used in this study's attempt to solve ordinary differential equations (Thomson, 1950).

7.1. SOLUTION FROM NEW TRANSFORM

Let's assume that equation (iii) holds true for the function f(t) where t and it must be continuous to exist.

First of all, finding New transform of some simple functions,

(i) Let f(t)= 1, then
$A(1)= \int_0^\infty {}^{-t}_{e} dt$ = $-[e^\infty - e^0]$= 1

(ii) F(t)=t then
$A(t)= \int_0^\infty (t)^{-t}_{e} dt$ on integration by parts gives
= 1

(iii) If f(t)=t^n then
$A(t^n) = \int_0^\infty (t)^{n}{}^{-t}_{e} dt = \frac{n!}{n+2}$ when n=0 then $A(t^0)$= A(1)=$\frac{n!}{2}$
When n=2 then $At^2=\frac{2!}{4}$

(iv) When f(t)=e^{at} then
$A(e^{at}) = \int_0^\infty e^{at}.e^{-t}dt=\int_0^\infty e^{at-t}\,dt=\frac{e^{\infty(a-1)}}{a-1} - \frac{e^{0(a-1)}}{a-1} = 0-\frac{e^0}{a01} = -\frac{1}{a-1} = \frac{1}{1-a}$

Solving the following transform: -

(a) $B(\sin bt) = \int_0^\infty e^{-t}\left(\frac{e^{ibt}}{2!} - \frac{e^{-ibt}}{2!}\right)\,dt = \frac{1}{2!}\int_0^\infty \left(e^{(ib-1)t} - e^{(-ib-1)t}\right)dt$

$= \frac{1}{2i}\left(\frac{1}{1-ib} - \frac{1}{1+ib}\right) = \frac{1}{2i}\left(\frac{1+ib-1+ib}{1+b^2}\right) = \frac{1}{2i}\left(\frac{2ib}{1+b^2}\right) = \frac{b}{1+b^2}$

(b) $B(\cos bt) = \int_0^\infty e^{-t}\left(\frac{e^{ibt}}{2} + \frac{e^{-ibt}}{2}\right)dt = \frac{1}{2}\int_0^\infty\left(e^{(ib-1)t} + e^{(-ib-1)t}\right)dt$

$= \frac{1}{2}\int_0^\infty \left(e^{(ib-1)t} + e^{-(1+ib)t}\right)dt = \frac{1}{2}\left(\frac{1}{1-ib} + \frac{1}{1+ib}\right) = \frac{1}{2}\left(\frac{1+ib+1-ib}{1+b^2}\right)$

$= \frac{1}{1+b^2}$, now in hyperbolic functions

(c) $B(\sinh bt) = \int_0^\infty e^{-t}\left(\frac{e^{bt}-e^{-bt}}{2}\right)dt = \frac{1}{2}\int_0^\infty\left(e^{(b-1)t} - e^{(-b-1)t}\right)dt$

$= \frac{1}{2}\left(\frac{1}{1-b} - \frac{1}{1+b}\right)dt = \frac{1}{2}\left(\frac{1+b-1+b}{1-b^2}\right) = \frac{1}{1-b^2}$

(d) $B(\cosh bt) = \int_0^\infty e^{-t}\left(\frac{e^{bt}+e^{-bt}}{2}\right)dt = \frac{1}{2}\int_0^\infty\left(e^{(b-1)t} + e^{(-b-1)t}\right)dt = \frac{1}{2}\left(\frac{1}{1-b} - \frac{1}{1+b}\right)dt$ $= \frac{1}{2}\left(\frac{1+b+1-b}{1-b^2}\right) = \frac{1}{1-b^2}$

Theorem 1.

Suppose N is the New transform of $[Af(t)] = N$, then the "New integral transform"

(i) $A[f't] = N - f(0)$

(ii) $A[f''t] = N - f(0) - f(0)$

(iii) $A[f^n(t)] = 1^n N - \sum_{N=0}^{n-1} \frac{f^N(0)}{-n-N}$

Proof

(i) $A[f'(t)] = \int_0^\infty (f'(t)e^{-t})\,dt$

Integrating by parts,

$A[f't] = N - f(0)$

(ii) $Af''(t) = \int_0^\infty f''(t)\,e^{-t}\,dt = N - f(0) - f(0)$ integrating by parts

(iii) Mathematical induction can give proof.

7.2. APPLICATIONS OF NEW TRANSFORM TO ORDINARY DIFFERENTIAL EQUATIONS

The applications of this transform in resolving ordinary differential equations are shown in the examples that follow. Using a formula,

$x' + p(x) = f(t)$ (vii)

with initial condition t = 0 or x (0) = a, where p and a are constants.

Now, applying this transform in equation (vii).

a x+ p x = f(t)

$\bar{X}$- f (0) +p(x) = f(t)= $\bar{X}$ $[p+1]$ = a+ f(t)

$\bar{X} = \frac{a+\overline{f(t)}}{1+p} = \frac{a+\overline{f(0)}}{1+p}$

Example 1. Suppose the first order differential equation

$\frac{dy}{dx} + y = 0,$ y(0) = 1

Solution:

Taking this transform to this equation

N(y) –Y(0)+A(y)=0

2AY=Y(0)

A(Y)=$\frac{1}{2}$Y(0)

Where NY is the New transform of function.

Example2. Suppose second order differential equation

y''+ y =0 , y(0)= y'(0) = 0

Taking this transform to this equation

A(Y)-Y'(0)- Y(0) +A(y)=0 or, $A(Y) = (1^2 + 1 = 2$

A(Y) = 1

Example 3. Let the differential equation $\frac{dx}{dt}$ + ax = 0 subject to condition x = 0 at t = 0.

Solution: To apply this transform to y = Nx tends to y- $x_0 + ay = 0$

Or, y = $\frac{x_0}{1+a}$ = N(x)

From transform table, $N^{-1}\frac{1}{1+a} = e^{-at}$

Accordingly, x = $x_0 e^{-at}$

Example 4. The solution of the equation

$\frac{d^2x}{dt^2}$+w^2x = $\cos wt$, t> 0 , subject to the condition that $\left.\begin{matrix} x = x_0 \\ \frac{dx}{dt} = x_1 \end{matrix}\right\}$ at x = 0.

Let y = N(x) then

N ($\cos wt$) = $\frac{1}{1^2+w^2}$

After transforming,

(Y-$x_o - x_1$) + $w^2 y = \frac{1}{1+w^2}$

Or, y = $\frac{x_o}{1+w^2}$+ $\frac{x_1}{1+w^2}$+ $\frac{1}{(1+w^2)^2}$,

From the table of transform,

x = x_o $\cos wt + x_1 \frac{\sin wt}{w} + {}^t/_{2w}$ $sinwt$ is the required solution.

7.3 APPLICATIONS OF THIS TRANSFORM TO THE SYSTEM OF DIFFERENTIAL EQUATIONS WITH CONTANT CO-EFFICIENT

Example 5. System of differential equations with constant co-efficient is given by

$$5\frac{dx_1}{dt}+4x_1+\frac{dx_2}{dt}=1, \quad t\geq 0 \quad \text{(viii)}$$

$$\frac{dx_1}{dt}+3x_2+4\frac{dx_2}{dt}=0 \quad \text{(ix)}$$

Subject to initial conditions

$$\left.\begin{matrix} x_1=0 \\ x_2=0 \end{matrix}\right\} \text{ at } t=0.$$

To solve these equations, suppose

$Nx_1 = y_1$

$Nx_2 = y_2$

$$\left.\begin{matrix} Nx_1 = y_1 \\ Nx_2 = y_2 \end{matrix}\right\} \quad \text{(x)}$$

As from relation $N\left(\frac{dx}{dt}\right)=x_0+y$ and $(3+2)y_1+y_2=1 \Rightarrow 5y_1+y_2=1$

$y_1+(4+3)y_2=0 \Rightarrow y_1+7y_2=0$

Solving these equations, $5(-7y_2)+y_2=1 \Rightarrow y_2=-\frac{1}{34}=Nx_2$

Using the transform values from transform table

$x_2=\frac{1}{5}\left(e^{-t}-e^{-\frac{6t}{11}}\right)$.

For y_1, $y_1=\frac{4+3}{(1+1)(11+6)}=Nx_1$

From transform values, $x_1=\frac{1}{2}-\frac{1}{5}e^{-t}-\frac{3}{10}e^{-\frac{6t}{11}}$

8. APPLICATIONS IN MODERN TECHNOLOGY

Integral transforms are foundational in modern technology, influencing various fields:

Signal Processing: Fourier and Wavelet transforms are crucial for filtering, compression, and reconstruction of signals.

Control Systems: Laplace transforms are integral in the design and analysis of control systems.

Quantum Mechanics: Fourier transforms are essential in solving the Schrödinger equation and in quantum field theory.

Medical Imaging: Techniques like MRI and CT scans rely on Fourier transforms for image reconstruction.

Communication Systems: Fourier and Laplace transforms are used in the analysis and design of communication systems.

9. NOVELTY AND RECENT ADVANCEMENTS

Recent advancements in integral transforms include the development of numerical algorithms for efficient computation, generalizations to multi-dimensional transforms, and applications in machine learning and data analysis. The advent of powerful computational tools has enabled the practical application of these transforms to complex real-world problems.

10. CONCLUSION AND RECOMMENDATIONS

This new transform is a powerful tool for solving linear ODEs, especially when initial conditions are provided. By transforming differential equations into algebraic ones, it simplifies the solution process, particularly for higher-order and system equations.

Future studies will examine other integral transformations, which are used to simplify problems in a variety of mathematical and scientific areas and evaluate functions across several domains.

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