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Discontinuity of functions and its some types

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Abstract

Continuity and discontinuity of functions is very important in mathematics. If a function is discontinuous at a point then it fails the definition of continuity of the function at that point and so the graph of the function has a break at that point. Such break of the graph of the function at that point may or may not be removed to make it continuous at that point. So we have different types of discontinuity of functions. I mention briefly about it below.

Key words: continuity/continuous, discontinuity/discontinuous, function, graph, removable and irremovable discontinuity, jump discontinuity and jump at a point, constant function, identity function etc.

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Introduction

A function f:[a, b] $\longrightarrow$ IR is said to be discontinuous at a point c $\in$ (a, b) if-

1- f is not continuous at the point c.

OR

2- At least one of the following 3 conditions hold:

i- $\lim_{x \to 0}$ f(x) does not exist.

ii- f(c) does not exist.

iii- $\lim_{x \to 0}$ f(x) $\neq$ f(c) though both exist.

The point c $\in$ (a, b), in this case, is called the point of discontinuity of the function f.

Note:- $\lim_{x \to 0}$ f(x) doesn't exist if,

i- Both f(c-) and f(c+) exist but are not equal.

ii- At least one of f(c-) and f(c+) doesn't exist.

Types of discontinuous

Various types of discontinuous of a function f:[a, b] $\longrightarrow$ IR at a point c $\in$ (a, b) are defined as follows.

i- Removable discontinuity :- If $\lim_{x \to c}$ f(x) exists but $\lim_{x \to c}$ f(x) $\neq$ f(c) ,i.e., if f(c-) and f(c+) both exist but f(c-) = f(c+) $\neq$ f(c) (f(c) may or may not exist) then f is said to have removable discontinuity at x = c. Such discontinuity can be removed by defining or redefining the function f at c.

ii- Irremovable discontinuity :- If a discontinuity of the function f at a point c cannot be removed then f is said to have irremovable discontinuity at the point x=c. The irremovable discontinuity is of following types.

I- Discontinuity of the first kind:- If f(c-) and f(c+) both exist but f(c-) $\neq$ f(c+), then f is said to have a discontinuity of the first kind at x=c.

This first kind discontinuity of x at c is also called a jump discontinuity of f at c and the non-zero number 'f(c+) - f(c-)' is called the jump of f at c.

II- Discontinuity of the first kind from left:- If f(c-) and f(c+) both exist but f(c-) $\neq$ f(c+) = f(c), then f is said to have a discontinuity of the first kind from the left at x=c.

It is also called the jump (or the left hand jump) discontinuity of f at c and the non-zero number 'f(c-) - f(c)' is called the left hand jump of f at c.

III- Discontinuity of the first kind from right:- If f(c-) and f(c+) both exist but f(c-) = f(c) ≠ f(c+), then f is said to have a discontinuity of the first kind from the right at x=c.

It is also called the jump (or the right hand jump) discontinuity of f at c and the non-zero number 'f(c+) - f(c)' is called the right hand jump of f at c.

IV- Discontinuity od the second kind:- If neither f(c-) nor f(c+) exists, then f is said to have a discontinuity of the second kind at x=c.

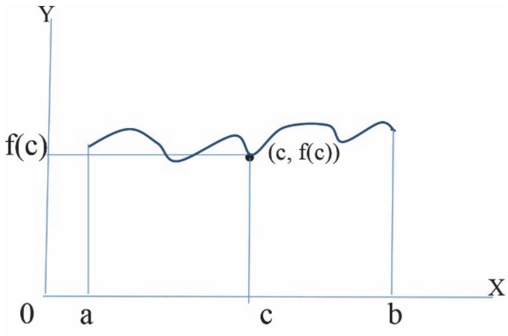
V- Discontinuity of the second kind from left:- If f(c-) does not exists, then f is said to have a discontinuity of the second kind from the left at x=c.

VI- Discontinuity of the second kind from right:- If f(c+) does not exists, then f is said to have a discontinuity of the second kind from the right at x=c.

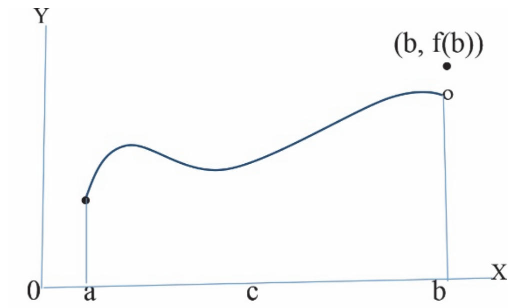
VII- Mixed Discontinuity:- If f has discontinuity of the second kind on one side (left or Right) of c and on the other side of c it may be continuous or may have discontinuity of the first kind, then f is said to have a mixed discontinuity at x=c.

VIII-Infinite Discontinuity:- If one or more of the functional limits: f(c+), f(c+), f(c-), f(c-) is +∞ or –∞ and f is discontinuous at c ; then f is said to have an infinite discontinuity at x=c. the point 'c' , in this case, is called a point of infinite discontinuity of f. Evidently, if f has discontinuity at x=c and it is unbounded in every neighborhood of c, then f will have an infinite discontinuity at c .

Some figures of continuity and discontinuity:

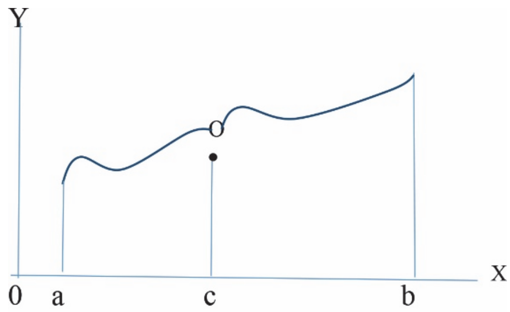


Here, f(c-) = f(c+) = f(c)
f is continuous at 'c'

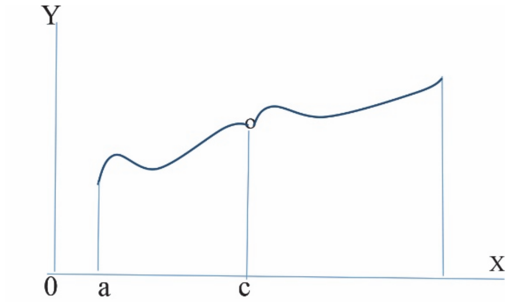


f(a+) = f(a)
f is cont. at 'a'

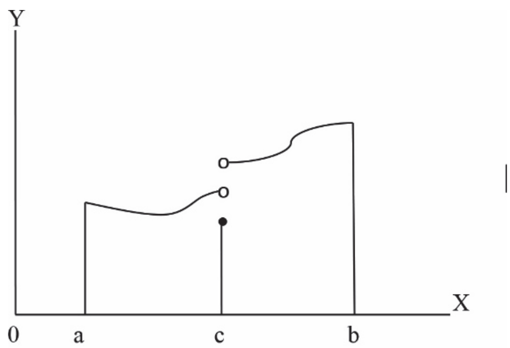
f(b-) ≠ f(b)
f is discont. at b



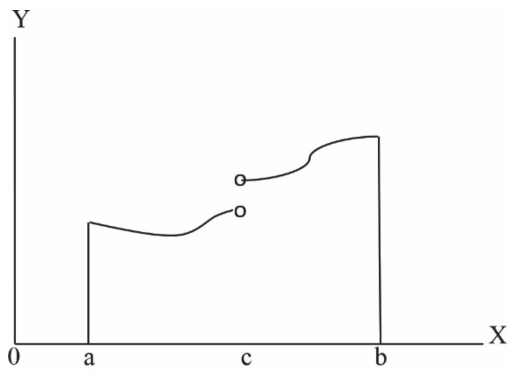
f(c-) = f(c+) ≠ f(c), f(c) is defined
f is removable discont. at c



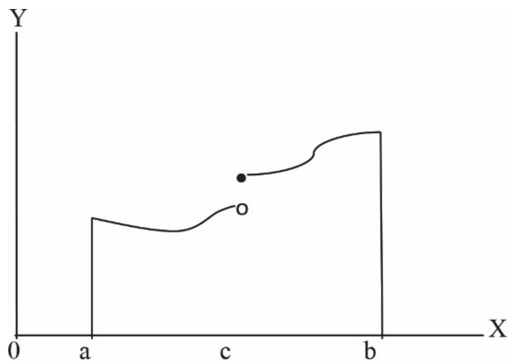
b f(c-) = f(c+) ≠ f(c), f(c) is not defined
f is removable discont. at c



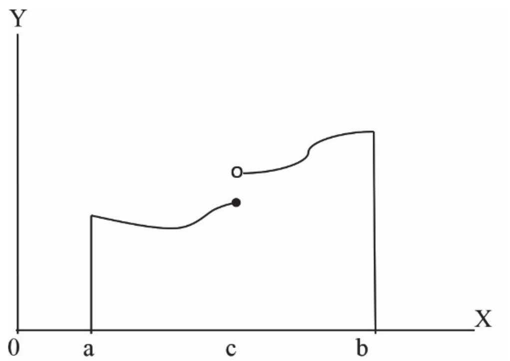
f(c-) ≠ f(c+) ≠ f(c), f(c) is defined
f is discont. at c (first kind discontinuity)



f(c-) ≠ f(c+), f(c) is not defined
f is discont. at c (first kind discontinuity)



f(c-) ≠ f(c) = f(c+)
f is cont. from the right at c
f is discont. of the first kind from the left at c
(Left hand jump of f at c)



f(c-) = f(c) ≠ f(c+)
f is cont. from the left at c
f is discont. of the first kind from the right at c
(Right hand jump of f at c)

<u>Some Examples</u>:

1) A constant function f: lR → lR defined by f(x) = k, a constant ∀x ϵlR is continuous on lR.

Proof: Clearly ∀C ϵlR and ∀ϵ >0 ∃ δ>0 such that ∀xϵlR, |x-c| <δ ⇒ |f(x) - f(c)|=|k-k|= 0< ϵ.

2) An identity function f:lR → lR defined by, f(x)= x, ∀x ϵlR is continuous on lR.

Clearly, $\forall$ cϵlR and $\forall$ ϵ>0 $\exists$ δ = ϵ>0 such that $\forall$ xϵlR, |x-c|< δ $\Rightarrow$ |f(x)-f(c)| = |x-c| < δ = ϵ.

3) A function f:lR → lR defined by, $f(x) = \begin{cases} X.\text{Sin}1/x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Is continuous at 0.

Solution: Here $f(0+) = \lim_{x\to 0+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} (0+h)\text{Sin}1/0+h$

$\lim_{h\to 0}$ = h.Sin1/h = 0 . k where k is a finite number lying between -1 and +1.

= 0

Similarly, $f(0-) = \lim_{x\to 0-} f(x) = \lim_{h\to 0} (0-h)\text{Sin}1/0-h = \lim_{h\to 0} h.\text{Sin}1/h = 0$ ($\because Sin(-\theta) = -Sin\theta$)

Given f(0) = 0.

Thus f(0+) = f(0-) = f(0). Therefore f is continues at 0.

4) A function f:lR → lR defined by, $f(x) = \begin{cases} |x|/x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

has a discontinuity of the first kind at x=0 or has a jump discontinuity at 0.

Solution: Here f(o+)= $\lim f(x) = \lim |x|/x = \lim x/x = 1$

and $f(0-) = \lim_{x\to 0-} f(x) = \lim_{x\to 0-} {}^{x-}|x|/x = \lim_{x\to 0-} -x/x = -1$

Given f(0) =0.

Thus f(0+) $\neq$ f(0-) $\neq$ f(0). Therefore f has a discontinuity of first kind at 0 or f has a jump discontinuity at 0 and the jump f at 0 = f(0+) – f(0-) =2.

5) A function f:lR → lR defined by, $f(x) = \begin{cases} \text{Sin}2x/x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

Is continuous at x $\neq$ 0.

At X = 0, $\lim_{x\to 0} f(x) = \lim_{x\to 0} \text{Sin}2x/x = \lim_{x\to 0} \text{Sin}2x/2x . 2 = 1 . 2 = 2 \neq 1 = f(0)$. f is discontinuous

at x = 0. Here, f has removable discontinuity at x = 0 as this discontinuity can be removed by redefining the function at x = 0, by f(0) = 2.

6) A function f:lR → lR defined by, $f(x) = \begin{cases} \frac{x-|x|}{x} & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases}$

Is discontinuous at x = 0.

Solution: $f(0-) = \lim_{x\to 0-} f(x) = \lim_{x\to 0-} \frac{x-|x|}{x} = \lim_{x\to 0-} \frac{x+x}{x} = 2$

and $f(0+) = \lim_{x\to 0+} f(x) = \lim_{x\to 0+} \frac{x-|x|}{x} = \lim_{x\to 0+} \frac{x-x}{x} = 1$. Given f(0) = 2.

Thus both f(0-) and f(0+) exist such that f(0) = f(0-) ≠ f(0+). Therefore f has discontinuity of the first kind from the right at x = 0.

7) The function f(x) defined by, $f(x) = \begin{cases} 2^{1/x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Is discontinuous at the origin.

Solution: $f(0-) = \lim_{x\to 0-} f(x) = \lim_{x\to 0-} 2^{1/x} = \lim_{h\to 0} 2^{1/0-h} = 2^{-\infty} = 0.$

And $f(0+) = \lim_{x\to 0+} f(x) = \lim_{x\to 0+} 2^{1/x} = \lim_{h\to 0} 2^{1/0+h} = 2^{+\infty} = \infty$. Given f(0) = 0.

Thus f(0) = f(0-) = 0 ≠ f(0+). Therefore the function is discontinuous at the origin and has an infinity discontinuity there.

3. Concluding Remark:-

From above we know that if a function f is discontinuous at a point "c" then the graph of the function has a break at the point (c, f(c)). In a removable discontinuity the break of the graph can be removed and it becomes continues at that point. It happens only when $\lim_{x\to c} f(x)$ exists and does not equal to f(c), if f(c) exists (f(c) may not be exists). If $\lim_{x\to c} f(x)$ does not exist, that is, at least one f(c-) and f(c+) does not exist or both exist but are not equal (f(c) may or may not exist) then f has an irremovable discontinuity at the point c and in this case the break of the graph of the function at the point (c, f(c)) cannot be removed.

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